AMPyC

Chapter 7: Stochastic Model Predictive Control II

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Recap: MPC for additive disturbances - Stochastic setting

Uncertain constrained system
$x(k+1) = f(x(k), u(k)) + w(k) \quad \Pr(x(k) \in \mathcal{X}) \geq p,\ \Pr(u(k) \in \mathcal{U}) \geq p, \qquad w(k) \sim \mathcal{Q}^w, i.i.d.$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$ **with given probability p**
2. Is 'stable': Converges to the origin **in a suitable sense**
3. Optimizes (nominal/**expected**) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Recap: Asymptotic Average Performance in Stochastic MPC

Asymptotic average performance bound

$$l_{\text{avg}} = \lim_{\bar{N}\to\infty} \frac{1}{\bar{N}} \sum_{k=0}^{\bar{N}-1} \mathbb{E}\left(l(x(k), \pi(x(k))\right) \leq C$$

Derive such a performance bound for stochastic MPC in three steps:

1. Lyapunov-like decrease implies asymptotic average performance bound

$$\mathbb{E}(V(x(k+1)) \,|\, x(k)) - V(x(k)) \leq -l(x(k), \pi(x(k))) + C \Rightarrow l_{\text{avg}} \leq C$$

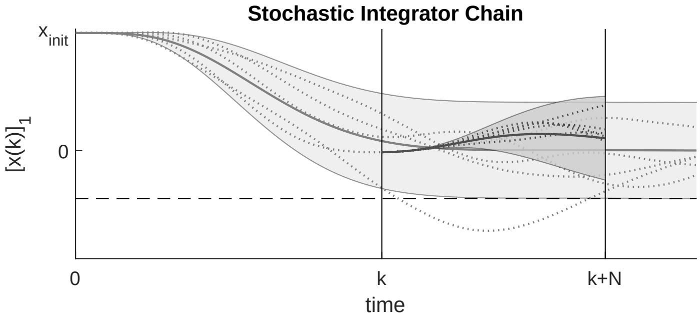
2. Apply this to the cost function decrease in MPC

$$\mathbb{E}(J^*(x(k+1) \,|\, x(k)) - J^*(x(k)) \leq -l(x(k), \pi(x)) + C$$

3. Tractable solution for (unconstrained/recursively feasible) linear stochastic MPC

Recap: Feasibility Issues in SMPC

Under general stochastic disturbances, feasibility can usually **not** be guaranteed



solid line: mean trajectory
dotted lines: trajectory samples
shaded area: confidence region

Recap: Feasibility Issues in SMPC

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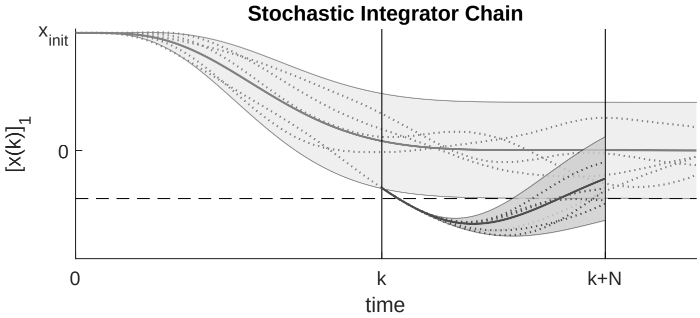

solid line:	mean trajectory
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Recap: Closed-loop chance constraint satisfaction

Closed-loop chance constraints:

$$(*)\;\; \Pr(x(k) \in \mathcal{X} \,|\, x(0)) \geq p, \;\; \forall k \geq 0$$

But MPC formulation successively enforces

$$(**)\;\; \Pr(x(k+1) \in \mathcal{X} \,|\, x(k)) \geq p, \;\; \forall k \geq 0$$

It can be easily seen that $(**) \Rightarrow (*)$ since

$$\Pr(x(k+1) \in \mathcal{X} \,|\, x(0)) = \int \underbrace{\Pr(x(k+1) \in \mathcal{X} \,|\, x(k))}_{\geq p} p(x(k) \,|\, x(0)) \,\mathrm{d}x(k) \geq p$$

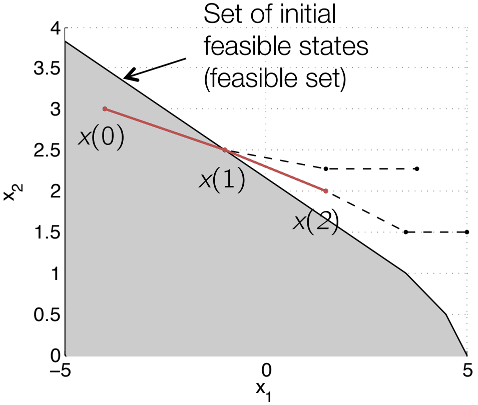
But it can be subject to considerable conservatism

Recap: Feasibility Issues in SMPC

Problem: (Potentially unbounded) stochastic disturbances can drive state initial state $x_0 = x(k)$ into infeasible region.

Several strategies to handle this problem

- Assume a maximum size of disturbance $\rightarrow$ use robust techniques
- Make use of recovery mechanisms
- Alternative forms of feedback



Recap: Feasibility Issues in SMPC

Problem: (Potentially unbounded) stochastic disturbances can drive state initial state $x_0 = x(k)$ into infeasible region.

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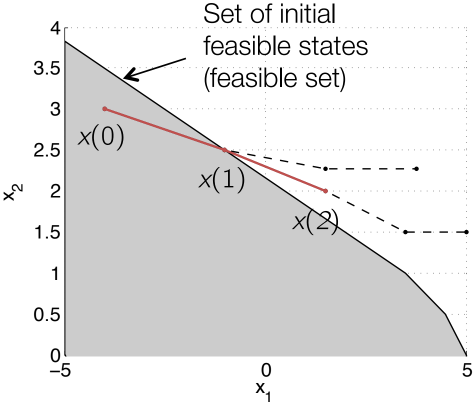

Recap: Stochastic MPC with Bounded Disturbances

Setup:

$$x(k+1) = Ax(k) + Bu(k) + w(k)$$

where $w(k) \sim \mathcal{Q}^w$ i.i.d. **and** all $w(k) \in \mathcal{W}$ with $\mathcal{W}$ a compact set.

- Stochastic disturbance $w(k)$ has **bounded support** $\mathcal{W}$
- Enables use of robust techniques for recursive feasibility

One possible approach related to "constraint-tightening" robust MPC:

- Enforce **chance** constraints w.r.t. **all** possible previous disturbances ($i-1$-steps robust, 1-step stochastic)
- Enforce terminal robust invariant set (within constraints) robustly
- For simplicity, we neglect input constraints for now, extension is straightforward
- References: [4, 5]

Recap: Stochastic MPC with Bounded Disturbances

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where $w(k) \sim \mathcal{Q}^w$ i.i.d. **and** all $w(k) \in \mathcal{W}$ with $\mathcal{W}$ a compact set.

- Stochastic disturbance $w(k)$ has **bounded support** $\mathcal{W}$
- Enables use of robust techniques for recursive feasibility

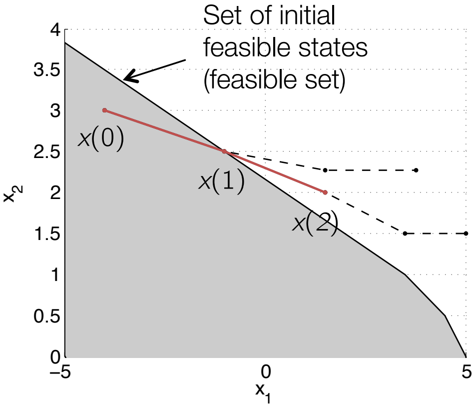
- Recursive feasibility properties follow directly from robust case
- Tractable in particular for polytopic $\mathcal{W}$ and half-space chance constraints (or polytopic constraint as collection of half-space chance constraints)
- Asymptotic average performance analysis directly applicable (candidate solution feasible)

Feasibility Issues in SMPC (This Chapter)

Problem: (Potentially unbounded) stochastic disturbances can drive state initial state $x_0 = x(k)$ into infeasible region.

Several strategies to handle this problem

- Assume a maximum size of disturbance $\rightarrow$ use robust techniques
- **Make use of recovery mechanisms**
- **Alternative forms of feedback**



Contents – 7: Stochastic MPC II

Understand different strategies of dealing with feasibility issues under unbounded noise, in particular:

- Recovery mechanisms
  - implications on stability
  - implications on closed-loop constraint satisfaction guarantees
- Indirect feedback
  - implications on stability
  - implications on closed-loop constraint satisfaction guarantees
  - practical implications

Outline

1. Stochastic MPC for unbounded disturbances with "Open-Loop MPC"

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

3. Stochastic MPC for unbounded disturbances with "Indirect Feedback"

Outline

1. Stochastic MPC for unbounded disturbances with "Open-Loop MPC"
2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"
3. Stochastic MPC for unbounded disturbances with "Indirect Feedback"

Recall: Chance Constrained Optimal Control

$$
\begin{aligned}
J^*(x) = \min_{\{u_k\}} \quad & \mathbb{E}\left(\sum_{i=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k) + w(k), \\
& u(k) = K(x(k) - \mathbb{E}(x(k))) + u_k, \\
& w(k) \sim \mathcal{N}(0, \Sigma_w), \text{ i.i.d.}, \\
& \Pr(h_{x,j}^\top x(k) \leq b_{x,j}) \geq p_{x,j} \; \forall j = 1, \ldots, N_{c,x}, \\
& \Pr(h_{u,j}^\top u(k) \leq b_{u,j}) \geq p_{u,j} \; \forall j = 1, \ldots, N_{c,u}, \\
& x(0) = x
\end{aligned}
$$

- (Approximate) solution to chance constrained stochastic optimal control problem
- Tractable in particular for Gaussian disturbances (unbounded support!)

Recall: Chance Constrained Optimal Control

$$
\begin{aligned}
\tilde{J}^*(x) = \min_{\{\bar{u}_k\}} \quad & \sum_{k=0}^{\bar{N}-1} \|\mathbb{E}(x(k))\|_Q^2 + \|\bar{u}_k\|_R^2 \\
\text{s.t.} \quad & \mathbb{E}(x(k+1)) = A\mathbb{E}(x(k)) + B\bar{u}_k, \\
& \mathrm{var}(x(k+1)) = (A+BK)\mathrm{var}(x(k))(A+BK)^\top + \Sigma_w, \\
& h_{x,j}^\top \mathbb{E}(x(k)) \leq b_{x,j} - \sqrt{h_{x,j}^\top \mathrm{var}(x(k)) h_{x,j}}\, \phi^{-1}(p_{x,j}) \ \forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top \bar{u}_k \leq b_{u,j} - \sqrt{h_{u,j}^\top K \mathrm{var}(x(k)) K^\top h_{u,j}}\, \phi^{-1}(p_{u,j}) \ \forall j = 1, \dots, N_{c,u}, \\
& \mathbb{E}(x(0)) = x, \ \mathrm{var}(x(0)) = 0
\end{aligned}
$$

- (Approximate) solution to chance constrained stochastic optimal control problem
- Tractable in particular for Gaussian disturbances (unbounded support!)

Recall: Chance Constrained Optimal Control

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\tilde{J}^*(x) = \min_{\{\bar{u}_k\}} \quad & \sum_{k=0}^{\bar{N}-1} \|\mathbb{E}(x(k))\|_Q^2 + \|\bar{u}_k\|_R^2 \\
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& \mathrm{var}(x(k+1)) = (A+BK)\mathrm{var}(x(k))(A+BK)^\top + \Sigma_w, \\
& h_{x,j}^\top \mathbb{E}(x(k)) \le b_{x,j} - \sqrt{h_{x,j}^\top \mathrm{var}(x(k)) h_{x,j}}\, \phi^{-1}(p_{x,j}) \; \forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top u_k \le b_{u,j} - \sqrt{h_{u,j}^\top K \mathrm{var}(x(k)) K^\top h_{u,j}}\, \phi^{-1}(p_{u,j}) \; \forall j = 1, \dots, N_{c,u}, \\
& \mathbb{E}(x(0)) = x, \; \mathrm{var}(x(0)) = 0
\end{aligned}
$$

Can we do this in receding horizon? Kind of..

Some Notation

Assumption: Prediction process is identical to "real" process

$$x(k+1) = Ax(k) + Bu(k) + w(k), \qquad w(k) \sim \mathcal{N}(0, \Sigma_w) \text{ i.i.d.}$$
$$x_{i+1} = Ax_i + Bu_i + w_i, \qquad w_i \sim \mathcal{N}(0, \Sigma_w) \text{ i.i.d.}$$

To simplify, we introduce some notation for the prediction dynamics:

$$\begin{aligned} \bar{x}_i &:= \mathbb{E}(x_i) \\ \bar{u}_i &:= \mathbb{E}(u_i) \\ d_i &:= x_i - \bar{x}_i \\ \bar{d}_i &:= \mathbb{E}(d_i) = 0 \\ \Sigma_i^x &:= \mathrm{var}(d_i) = \mathrm{var}(x_i) \end{aligned} \qquad \text{resulting in } \rightarrow \qquad \begin{aligned} u_i &= Kd_i + \bar{u}_i \\ \bar{x}_{i+1} &= A\bar{x}_i + B\bar{u}_i \\ \Sigma_{i+1}^x &= (A + BK)\Sigma_i^x (A + BK)^\mathsf{T} + \Sigma_w \end{aligned}$$

where the expectations are understood conditioned on the initial state of the prediction $i = 0$

Intuition: "Open-Loop" Stochastic MPC

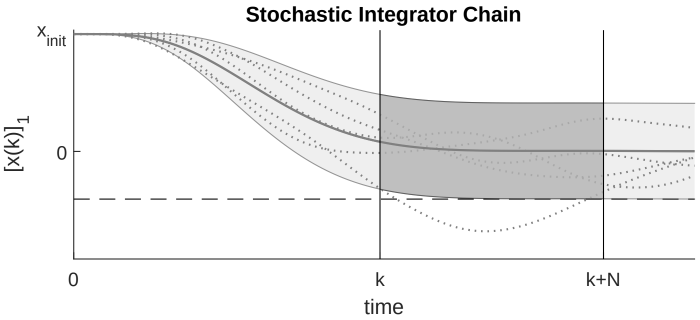
$$
\begin{aligned}
\tilde{J}^*(x(k)) = \min_{\{\bar{u}_i\}} \quad & \|\bar{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 \\
\text{s.t.} \quad & \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \\
& \Sigma_{i+1}^x = (A+BK)\Sigma_i^x(A+BK)^\top + \Sigma_w, \\
& h_{x,j}^\top \bar{x}_i \leq b_{x,j} - \sqrt{h_{x,j}^\top \Sigma_i^x h_{x,j}}\, \phi^{-1}(p_{x,j}) \ \forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top \bar{u}_i \leq b_{u,j} - \sqrt{h_{u,j}^\top K \Sigma_i^x K^\top h_{u,j}}\, \phi^{-1}(p_{u,j}) \ \forall j = 1, \dots, N_{c,u}, \\
& \bar{x}_N \in \bar{\mathcal{X}}_f, \\
& \bar{x}_0 = \bar{x}_{1|k-1}, \ \Sigma_0^x = \Sigma_{1|k-1}^x
\end{aligned}
$$

- Apply $u(k) = K(x(k) - \bar{x}_0^*) + \bar{u}_0^*$
- Initialize at predicted distribution

$\rightarrow$ Nominal MPC: recursively feasible[1] and satisfies closed-loop chance constraints

[1] Requires typical selection of terminal set for nominal MPC (within maximum tightened constraints)

Illustration: "Open-Loop" Stochastic MPC



solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

- Original optimization problem (Lecture 6) solves for entire task horizon $\bar{N}$
- "Open-Loop" stochastic MPC repeatedly solves for shortened horizon N
- Equivalent (given right terminal cost & set, sufficiently long horizon)

$\rightarrow$ **This "MPC" reduces to iterative trajectory planning method (no feedback from $x(k)$)**

Outline

1. Stochastic MPC for unbounded disturbances with "Open-Loop MPC"

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

3. Stochastic MPC for unbounded disturbances with "Indirect Feedback"

Outline

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

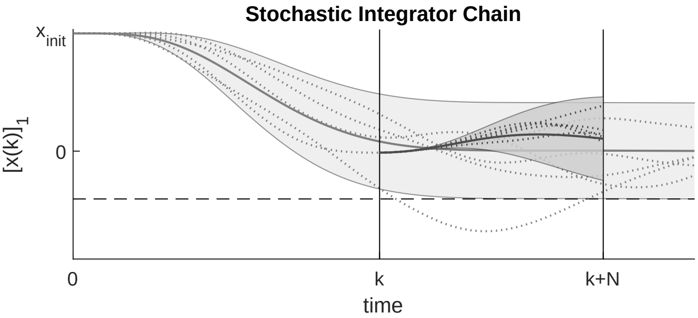
   Recovery Initialization

   Implications for Stability

   Implications for Chance Constraint Satisfaction

   Closed-loop Chance Constraint Satisfaction with Probabilistic Reachable Sets

Idea: Introduce Feedback whenever Feasible



solid line: mean trajectory
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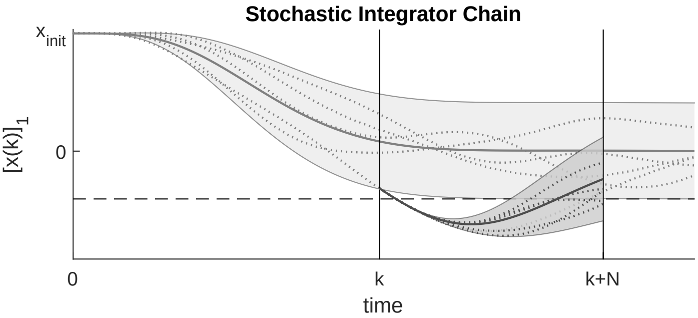
shaded area: confidence region

Recovery Initialization

Case 1: $\bar{x}_0 = x(k)$, $\Sigma_0^x = 0$ (Whenever feasible)

Case 2: $\bar{x}_0 = \bar{x}_{1|k-1}$, $\Sigma_0^x = \Sigma_{1|k-1}^x$ (Otherwise, guaranteed feasible)

Idea: Introduce Feedback whenever Feasible



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Recovery Initialization

Case 1: $\bar{x}_0 = x(k)$, $\Sigma_0^x = 0$ (Whenever feasible)

Case 2: $\bar{x}_0 = \bar{x}_{1|k-1}$, $\Sigma_0^x = \Sigma_{1|k-1}^x$ (Otherwise, guaranteed feasible)

Stochastic MPC with Recovery Initialization

Aims to combine "best of both worlds":

- Feasibility guarantees from "open-loop" stochastic MPC
- Use feedback whenever possible

But theoretical analysis proves challenging!

Outlook

$\rightarrow$ What are the implications for stability/performance?

- Candidate solution does not remain feasible (from measured state)
- Cost from measured state may increase (over "shifted" solution) [1]

$\rightarrow$ What are implications for closed-loop constraint satisfaction?

- No direct guarantees on closed-loop (as opposed to bounded disturbance case) [3]
- Can also be conservative! (same for bounded disturbance case) [5]

Outline

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

    Recovery Initialization

    Implications for Stability

    Implications for Chance Constraint Satisfaction

    Closed-loop Chance Constraint Satisfaction with Probabilistic Reachable Sets

Stability under Recovery Initialization

Recovery Initialization

Case 1 (C_1):	$\bar{x}_0 = x(k)$,	$\Sigma_0^x = 0$	(Whenever feasible)
Case 2 (C_2):	$\bar{x}_0 = \bar{x}_{1\|k-1}$,	$\Sigma_0^x = \Sigma_{1\|k-1}^x$	(Otherwise, guaranteed feasible)

For asymptotic average performance bound, we want

$$\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k)) - J^*(\tilde{x}(k)) \leq -l(x(k), \pi(\tilde{x}(k)) + C$$

(Note that this is not a state-feedback controller: extended state $\tilde{x}(k) = (x(k), \bar{x}_{1|k-1}, \Sigma_{1|k-1}^x))$

We need to show this considering both possible cases (C_1 and C_2)!

$$\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k)) = \mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k), C_1)\, \Pr(C_1 \,|\, \tilde{x}(k)) + \mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k), C_2)\, \Pr(C_2 \,|\, \tilde{x}(k))$$

Case 2: Recovery Initialization

Given Case 2, we have

$$\bar{x}_0 = \bar{x}_{1|k-1},\ \Sigma_0^x = \Sigma_{1|k-1}^x$$

This implies $x_0 \stackrel{d}{=} x_1^*$, in fact the candidate sequence

$$\bar{U} = \{\bar{u}_0, \dots, \bar{u}_{N-1}\} = \{\bar{u}_1^*, \dots, \bar{u}_{N-1}^*, K\bar{x}_N^*\}$$

remains feasible (shifting the previous solution works), resulting in

$$\{x_0, \dots, x_N\} \stackrel{d}{=} \{x_1^*, \dots, x_{N-1}^*, (A + BK)x_N^* + w)\} \qquad \text{(equal in distribution)}$$

Choosing $P = (A + BK)^\top P(A + BK) + Q + K^\top RK$ and following the previous lecture we get

$$\boxed{\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k), C_2) - J^*(\tilde{x}(k)) \leq -l(x(k), \pi(\tilde{x}(k)) + C}$$

Case 1: Feedback Initialization

Given Case 1, we have

$$\bar{x}_0 = x(k),\ \Sigma_0^x = 0$$

and we know a feasible solution exists from this initial condition.

Problem: This does not mean that the candidate sequence is feasible!
$\rightarrow$ shifting the previous solution **does not** work

In fact, it may happen that

$$J^*(\tilde{x}(k+1)) > \bar{J}(\tilde{x}(k+1))$$

(feasible (& optimal) solution with higher cost $J^*(\tilde{x}(k+1))$ exists)

Possible remedy [1,2]: Use feedback (C_1) only if $J^*(\tilde{x}(k+1)) \leq \bar{J}(\tilde{x}(k+1))$

$$\mathbb{E}(J^*(\tilde{x}(k+1)) \,|\, \tilde{x}(k), C_1) - J^*(\tilde{x}(k)) \leq -l(x(k), \pi(\tilde{x}(k)) + C$$

Stability-Adapted Recovery Scheme [1,2]

Adapted Recovery Initialization

Case 1 (C_1): $\bar{x}_0 = x(k)$, $\Sigma_0^x = 0$ (If feasible & reduces cost)

Case 2 (C_2): $\bar{x}_0 = \bar{x}_{1|k-1}$, $\Sigma_0^x = \Sigma_{1|k-1}^x$ (Otherwise, guaranteed feasible & cost decrease)

With this we have

$$\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k)) - J^*(\tilde{x}(k)) \leq -l(x(k), \pi(\tilde{x}(k)) + C$$

since

$$\begin{aligned}\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k)) - J^*(\tilde{x}(k)) = & \underbrace{\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k), C_1) - J^*(\tilde{x}(k))}_{\leq -l(x(k), \pi(\tilde{x}(k))+C} \Pr(C_1 \,|\, \tilde{x}(k)) \\ & + \underbrace{\mathbb{E}(J^*(\tilde{x}(k+1) \,|\, \tilde{x}(k), C_2) - J^*(\tilde{x}(k))}_{\leq -l(x(k), \pi(\tilde{x}(k))+C} \Pr(C_2 \,|\, \tilde{x}(k))\end{aligned}$$

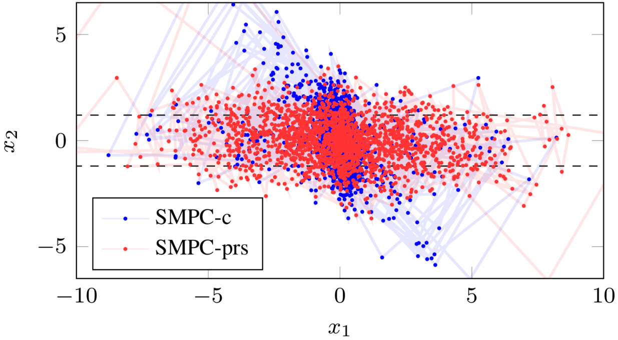
Effect of Reduced Feedback [3]

Scenario

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k) + w(k)$$

Rare (unmodeled) large disturbances

- Red: C_1 whenever feasible
- Blue: C_1 feasible & cost decreases



When driven far away from the origin blue keeps candidate solution
→ no feedback on (constrained) optimization problem
→ applies mainly tube controller, ignoring constraints

Outline

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

    Recovery Initialization

    Implications for Stability

    Implications for Chance Constraint Satisfaction

    Closed-loop Chance Constraint Satisfaction with Probabilistic Reachable Sets

Constraint Satisfaction from Recursive Feasibility

Closed-loop chance constraints

$$(*)\ \ \Pr(x(k) \in \mathcal{X} \mid x(0)) \geq p,\ \forall k \geq 0$$

If feasible: MPC formulation enforces (since $x_1 \stackrel{d}{=} x(k+1)$ given $x(k)$)

$$(**)\ \ \Pr(x(k+1) \in \mathcal{X} \mid x(k)) \geq p,\ \forall k \geq 0$$

Given feasibility at all $k \geq 0$ we have that $(**) \Rightarrow (*)$ since

$$\Pr(x(k) \in \mathcal{X} \mid x(0)) = \int \underbrace{\Pr(x(k) \in \mathcal{X} \mid x(k-1))}_{\geq p} p(x(k-1) \mid x(0))\, \mathrm{d}x(k-1) \geq p$$

Constraint Satisfaction from Recursive Feasibility

Closed-loop chance constraints

$$(*)\ \ \Pr(x(k) \in \mathcal{X} \mid x(0)) \geq p,\ \forall k \geq 0$$

If feasible: MPC formulation enforces (since $x_1 \stackrel{d}{=} x(k+1)$ given $x(k)$)

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Given feasibility at all $k \geq 0$ we have that $(**) \Rightarrow (*)$ since

$$\Pr(x(k) \in \mathcal{X} \mid x(0)) = \int \underbrace{\Pr(x(k) \in \mathcal{X} \mid x(k-1))}_{\geq p} p(x(k-1) \mid x(0))\, \mathrm{d}x(k-1) \geq p$$

Now we cannot assume feasibility in each time step (cannot assume $(**)$ holds):

$$\boxed{\Pr(x(k+1) \in \mathcal{X} \mid x(k)) = \underbrace{\Pr(x(k+1) \in \mathcal{X} \mid x(k), C_1)}_{\geq p} \Pr(C_1) + \underbrace{\Pr(x(k+1) \in \mathcal{X} \mid x(k), C_2)}_{\text{?? can be } \ll p} \Pr(C_2)}$$

Counterexample: Closed-loop Constraint Violation I

Double integrator system:

$$x(k+1) = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} x(k) + \underbrace{\begin{bmatrix}0.5\\ 1\end{bmatrix}}_{B} u(k) + w(k)$$

with i.i.d. Gaussian disturbance

$$w(k) \sim \mathcal{N}(0, BB^\mathsf{T})$$

Velocity chance constraint

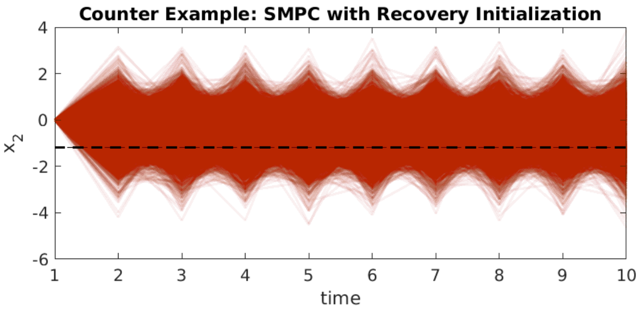
$$\Pr([x(k)]_2 \geq -1.2) \geq 0.8$$

Controlled with MPC

$$\begin{aligned}
\min_{\{\bar{u}_i\}} \quad & \|\bar{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 \\
\text{s.t.} \quad & \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \\
& \Sigma_{i+1}^x = (A+BK)\Sigma_i^x(A+BK)^\mathsf{T} + \Sigma_w, \\
& [\bar{x}_i]_2 \geq -1.2 + \sqrt{[\Sigma_i^x]_{2,2}}\,\phi^{-1}(0.8), \\
& \bar{x}_N \in \mathcal{X}_f, \\
& \bar{x}_0 = \begin{cases} x(k),\ \Sigma_0^x = 0 & \text{feasible \& cost decrease} \\ \bar{x}_0 = \bar{x}_{1|k-1},\ \Sigma_0^x = \Sigma_{1|k-1}^x, & \text{otherwise} \end{cases}
\end{aligned}$$

Applied input: $u(k) = K(x(k) - \bar{x}_0^*) + u_0^*$

Counterexample: Closed-loop Constraint Violation II



- Optimal to minimize $[x(k)]_2$ as much as possible
- Given $[x(k)]_2 \ll -1.2$: MPC may be infeasible and tube controller is applied (recovery)
- Given $[x(k)]_2 \gg -1.2$: MPC aggressively drives system back into constraint (cost optimal)

Closed-loop system more 'aggressive' than open-loop plan, constraint violation: 24.1% > 20%

Outline

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

    Recovery Initialization

    Implications for Stability

    Implications for Chance Constraint Satisfaction

    Closed-loop Chance Constraint Satisfaction with Probabilistic Reachable Sets

Stochastic MPC with Probabilistic Reachable Sets

Problem: The distribution of closed-loop $x(k)$ is very hard to establish.
Define $x(k) = \bar{x}_{0|k} + d(k)$. Then

- $d(k) = 0$ whenever MPC feasible, otherwise $d(k+1) = (A + BK)d(k) + w(k)$
- depends on MPC optimization

$\rightarrow$ hard to derive constraint tightening that applies in closed-loop

Idea: We can analyze the distribution of $d_{k+1} = (A + BK)d_k + w_k$. Can we relate $d(k)$ to d_k, for $d(0) = d_0 = 0$?

General closed-loop constraint satisfaction is an open problem:
We will discuss a result for the special case of unimodal $w(k)$ and symmetric constraint tightening.

Convex Unimodal Random Variable

A random variable w is convex unimodal if for every direction $\hat{w} \in \mathbb{R}^n$ and every convex set $\mathcal{F}$ symmetric around the origin the probability $\Pr(w + c\hat{w} \in \mathcal{F})$ is non-increasing in $c \geq 0$.

Symmetric Tightening with Unimodal Disturbances

[3, Theorem 3]

Let $w(k)$ and w_k be i.i.d. convex unimodal random variables and $\mathcal{F}$ a convex symmetric set around the origin. Let $d_{k+1} = (A+BK)d_k + w_k$, $d_0 = 0$ be the (linearly) predicted deviation, and $d(k) = x(k) - \bar{x}_{0|k}$ the actual deviation under stochastic MPC with recovery initialization. Then

$$\Pr(d(k) \in \mathcal{F}) \geq \Pr(d_k \in \mathcal{F}) \; \forall k \geq 0$$

Proof idea:

$$\begin{aligned}
&\Pr((A+BK)d(k) + w(k) \in \mathcal{F}) \\
&= \Pr((A+BK)d(k) + w(k) \in \mathcal{F} \mid C_1)\Pr(C_1) + \Pr((A+BK)d(k) + w(k) \in \mathcal{F} \mid C_2)\Pr(C_2) \\
&= \Pr(w(k) \in \mathcal{F} \mid C_1)\Pr(C_1) + \Pr((A+BK)d_k + w(k) \in \mathcal{F} \mid C_2)\Pr(C_2) \\
&\geq \Pr((A+BK)d(k) + w(k) \in \mathcal{F} \mid C_1)\Pr(C_1) + \Pr((A+BK)d_k + w(k) \in \mathcal{F} \mid C_2)\Pr(C_2) \\
&= \Pr((A+BK)d_k + w_k \in \mathcal{F})
\end{aligned}$$

(due to unimodality of $w(k)$ and symmetry of $\mathcal{F}$)

Probabilistic Reachable Sets

This result can be used to derive sets for constraint tightening that hold in closed-loop:

Probabilistic Reachable Set [3, 4]

A set $\mathcal{F}_i^p$ is an i-step probabilistic reachable set (PRS) of probability level p for process $\{d(k)\}$ initialized at $d(0)$ if

$$\Pr(d(i) \in \mathcal{F}_i^p \mid d(0)) \geq p$$

Reformulation of chance constraints with $x(k) = \bar{x}_0 + d(k)$ and $\mathcal{F}_k^p$ PRS for $d(k)$.

$$\Pr(x(k) \in \mathcal{X}) \geq p \Leftarrow \bar{x}_0 \in \mathcal{X} \ominus \mathcal{F}_k^p$$

([3,Theorem 3]) PRS for $d_{k+1} = (A + BK)d_k + w_k$ is also PRS for $d(k) \to$ tractable computation

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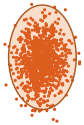

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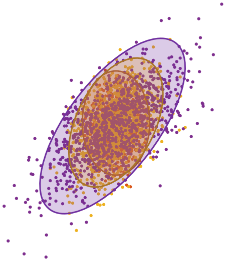

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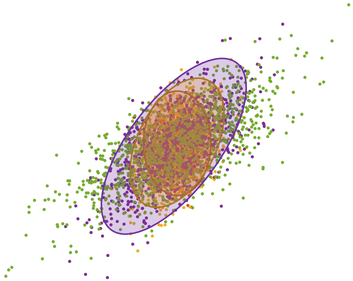

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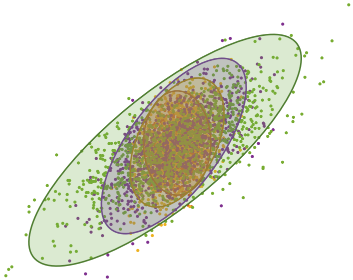

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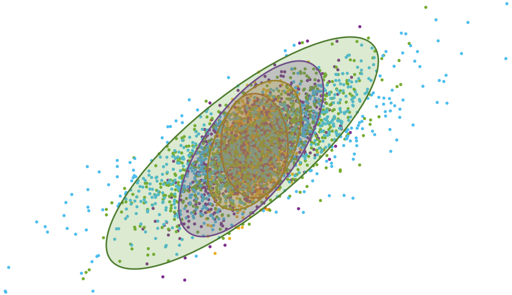

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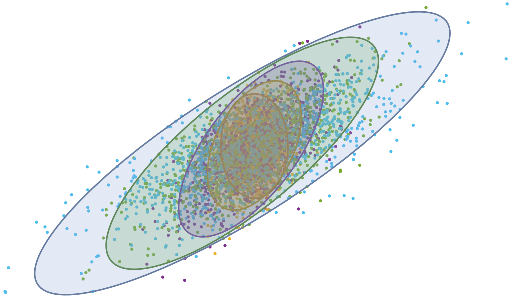

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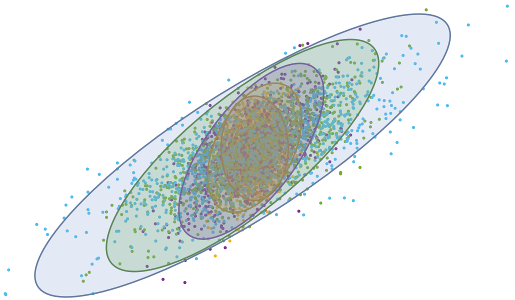

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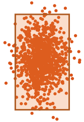

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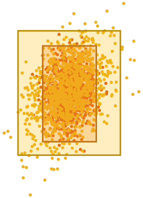

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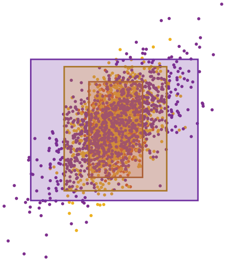

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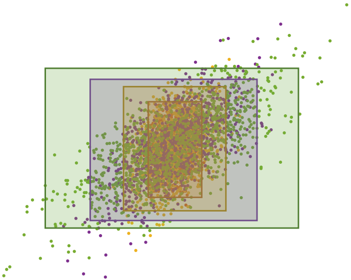

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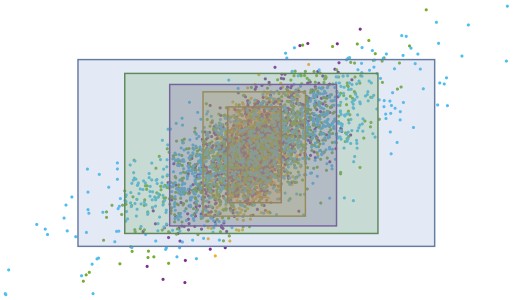

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SMPC with PRS & Recovery Initialization

$$
\begin{aligned}
\min_{\{\bar{u}_i\}} \quad & \|\bar{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 \\
\text{s.t.} \quad & \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \\
& \bar{x}_i \in \mathcal{X} \ominus \mathcal{F}_{i+k}^p, \\
& \bar{x}_N \in \mathcal{Z}_f \subset \mathcal{X} \ominus \mathcal{F}_\infty^p, \\
& \bar{x}_0 = \begin{cases} x(k), & \text{if feasible} \\ \bar{x}_{1|k-1}, & \text{otherwise} \end{cases}
\end{aligned}
$$

Closed-loop constraint satisfaction guarantee if $w(k)$ unimodal and $\mathcal{F}_i^p$ symmetric

Summary

- Result for tightening with convex symmetric probabilistic reachable sets
- Alternatives exists (e.g. soft constraints) but analysis similarly lacking

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1. Stochastic MPC for unbounded disturbances with "Open-Loop MPC"

2. Stochastic MPC for unbounded disturbances with "Recovery Initialization"

3. Stochastic MPC for unbounded disturbances with "Indirect Feedback"

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   Constraint Satisfaction in Prediction

Chance Constraints in Closed-Loop vs. in Prediction

We want to show:

$$(*)\ Pr(x(k) \in \mathcal{X} \,|\, x(0)) \quad \text{(conditioned on initial state } x(0))$$

But typically enforce

$$(**)\ Pr(x_i \in \mathcal{X} \,|\, x_0 = x(k)) \quad \text{(conditioned on measured state } x(k))$$

Goal was to guarantee that $(**)$ always holds, and then $(**) \Rightarrow (*)$

- Constraint tightening SMPC for bounded disturbances (last lecture)

In general, enforcing $(**)$ for all k is much stricter than $(*)$

- Leads to feasibility issues
- Can be conservative, if $(*)$ is the required condition

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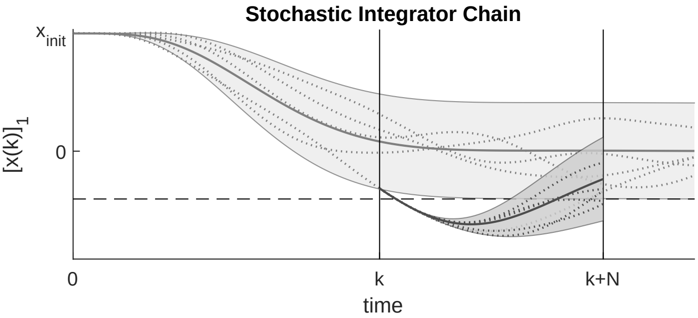

solid line:	mean trajectory
dotted lines:	trajectory samples
shaded area:	confidence region

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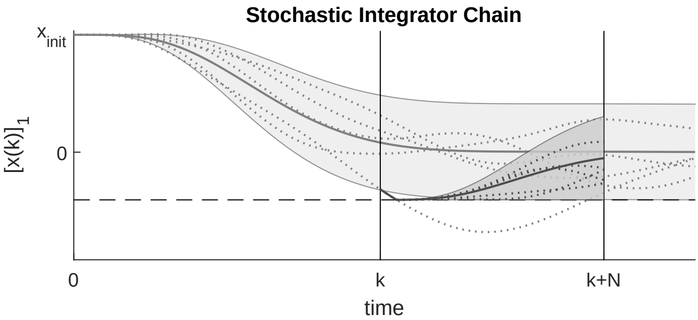

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Chance Constraints in Closed-Loop vs. in Prediction

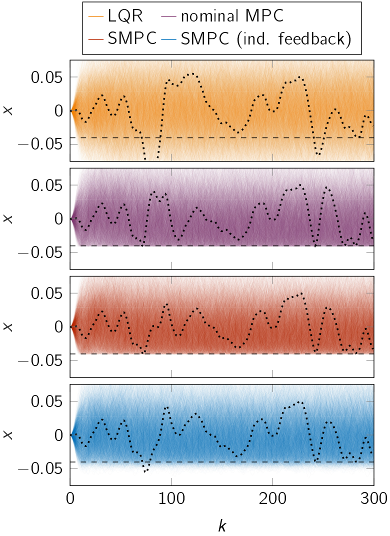
Third order integrator chain:

$$x(k+1) = \begin{bmatrix} 1 & 0.1 & 0.1^2/2 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.1^3/6 \\ 0.1^2/2 \\ 0.1 \end{bmatrix} (u(k) + w(k))$$

with i.i.d. $w(k) \sim \mathcal{N}(0,1)$ and chance constraint

$$\Pr\left([x]_1 \geq -\sqrt{[\Sigma_\infty]_{1,1}}\right) \geq 0.84$$

- Unconstrained (LQR) solution satisfies constraint
- Main effect of disturbance on $[x]_1$ is delayed (needs to propagate through system)
- SMPC results in virtually no "constraint softening"
- Side effect: Aggressive control inputs to ensure feasibility



Indirect feedback SMPC: Idea

Introduce **nominal state** $z(k)$ and let it evolve according to **nominal input** $v(k) = v_0^*$

$$\begin{aligned} z(k+1) &= Az(k) + Bv(k) \\ x(k+1) &= Ax(k) + Bu(k) + w(k) = Ax(k) + B(K(x(k) - z(k)) + v(k)) + w(k) \\ e(k+1) &= x(k+1) - z(k+1) = (A + BK)e(k) + w(k) \end{aligned}$$

$\rightarrow$ no **direct feedback** from measurement $x(k)$ on $z(k)$
$\rightarrow$ error state $e(k) = x(k) - z(k)$ evolves linearly and independent of $v(k)$

Straightforward to formulate constraints on closed-loop (similar to 'open-loop' stochastic MPC)

$$z(k) \in \mathcal{X} \ominus \mathcal{F}_k^p \Rightarrow \Pr(x(k) \in \mathcal{X}) \geq p$$

As opposed to "open-loop" stochastic MPC, we nevertheless introduce feedback by optimizing over cost given measured state $x(k)$, i.e. $x_0 = x(k)$ (**indirect feedback**)

$$\min \mathbb{E}\left(\|x_N\|_P^2 + \sum_{i=0}^{N-1} \|x_i\|_Q^2 + \|u_i\|_R^2 \,\middle|\, x_0 = x(k) \right)$$

Indirect Feedback SMPC: Resulting Formulation

$$
\begin{aligned}
\tilde{J}^*(x(k)) = \min_{\{v_i\}} \quad & \|\bar{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 \\
\text{s.t.} \quad & \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \\
& z_{i+1} = Az_i + Bv_i, \\
& \bar{e}_i = \bar{x}_i - z_i, \\
& \bar{u}_i = K\bar{e}_i + v_i, \\
& z_i \in \mathcal{X} \ominus \mathcal{F}_{k+i}^p, \\
& z_N \in \mathcal{Z}_f \subset \mathcal{X} \ominus \mathcal{F}_\infty^p, \\
& \bar{x}_0 = x(k),\ z_0 = z(k) = z_{1|k-1}
\end{aligned}
$$

- Applied input: $u(k) = K(x(k) - z(k)) + v_0^*$
- $\mathcal{Z}_f$ terminal (nominal) invariant set
- $\mathcal{F}_{k+i}^p$ probabilistic reachable sets for $e(k+i)$

Indirect Feedback SMPC: Main Properties

Candidate sequence remains feasible (details in recitation)

$$\bar{V} = \{v_1^*, \ldots, v_{N-1}^*, \pi_f(z_N^*)\}$$
$$\rightarrow \bar{Z} = \{z_1^*, \ldots, z_N^*, Az_N^* + B\pi_f(z_N^*)\}$$

From this follows recursive feasibility and

- Closed-loop chance constraint satisfaction, since

$$x(k) = z(k) + e(k),\ z(k) \in \mathcal{X} \ominus \mathcal{F}_k^p \text{ and } \Pr(e(k) \in \mathcal{F}_k^p) \geq p$$

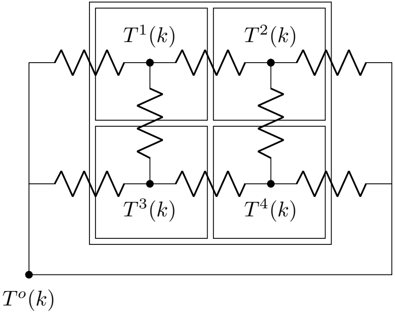
- Asymptotic average performance bound

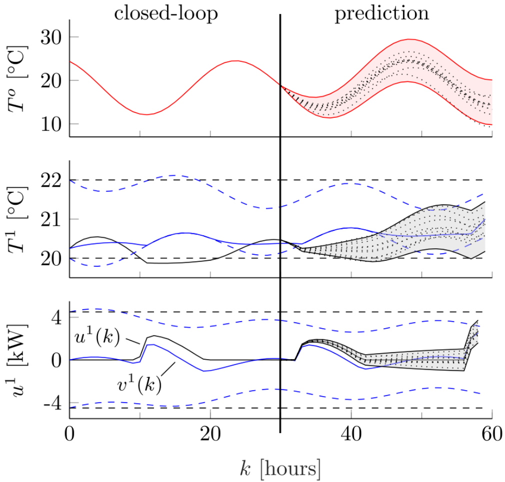
$$\lim_{\bar{N}\to\infty} \frac{1}{\bar{N}} \sum_{k=0}^{\bar{N}-1} \mathbb{E}(l(x(k), u(k))) \leq \mathrm{tr}(P\Sigma_w)$$

when choosing $\pi_f(x) = Kx$ and $P = (A + BK)^\top P(A + BK) + Q + K^\top RK$

Indirect Feedback SMPC: Building Control

- Modeled as resistance network (linear)
- Heating of 4 different Rooms
- Comfort constraint $\Pr(T_j \in [20, 22]) \geq 0.9$
- Sparsity inducing input cost $\|u\|_1$ (1-norm)





Indirect Feedback SMPC: Practical Effects

Standard double integrator system:

$$x_{i+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_i + \underbrace{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}_{B} u_i + w_i, \;\; w_i \sim \mathcal{N}(0, BB^\top)$$

Velocity chance constraint $\Pr([x(k)]_2 \leq 3) \geq 0.9$

But now: **Model mismatch** $B_{\text{sim}} = \frac{1}{5} B$

Controller	Closed-Loop Cost	Constraint Violation
Nominal MPC	130	21.7%
(direct feedback) SMPC	145	11.0%
(indirect feedback) SMPC	91	19.8%

Summary

- Stochastic MPC remains active research topic, in particular
    - unbounded disturbance distributions (this lecture)
    - nonlinear stochastic MPC (not discussed)

- Common formulation (satisfying chance constraints in prediction) may need critical re-evaluation
    - Indirect feedback as possible alternative
    - Connection to stochastic reference governors

References and further reading

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