AMPyC

Chapter 3: Nonlinear robust MPC - Part I

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Robust MPC - motivation

- Previous chapters: “Inherent robustness of nominal MPC”
  - Continuity $\Rightarrow$ (Local/regional) Input to state stability (ISS) w.r.t. (small enough) disturbances
  - Qualitative robustness results of nominal implementation
  - Soft-constraints may alleviate some of the restrictions

- What is missing:
  - Clear guaranteed region of attraction
  - Robust stability (quantitative performance bounds by design)
  - Robust satisfaction of state constraints (safety)

- Goal:
  - Develop MPC scheme with “robustness by design”
  - Exploit bounds on disturbance in the design

Contents - 3: Robust MPC

- Recall linear robust MPC
  - Understand concept of tubes and nominal dynamics
  - Robust MPC approaches based on constraint tightening and invariant sets
  - Prove closed-loop properties (recursive feasibility, stability, constraint satisfaction)

- Nonlinear tube-MPC
  - Investigate error dynamics for nonlinear systems
  - Learn concept of incremental stability
  - Learn to design a tube-MPC for nonlinear systems
  - Prove closed-loop properties (recursive feasibility, stability, constraint satisfaction)

Outline

1. Linear robust MPC

2. Nonlinear systems and incremental stability

3. Nonlinear tube-MPC

Outline

1. Linear robust MPC

    Error dynamics, reachable and invariant sets

    Constraint-tightening MPC

    Tube-MPC

Linear robust MPC

Uncertain constrained linear system

$$x(k+1) = Ax(k) + Bu(k) + w(k), \quad (x(k), u(k)) \in \mathcal{X} \times \mathcal{U}, \quad w(k) \in \mathcal{W}.$$

Main idea:
Separate the true system (x, u) into a 'nominal' system (z, v) and error $e = x - z$.

1. Drive 'nominal' (disturbance free) system to the origin
   $z(k+1) = Az(k) + Bv(k)$
2. Ensure that the real trajectory "tracks" the nominal trajectory by using an additional feedback
   $u(k) = K(x(k) - z(k)) + v(k)$

Only optimize and predict over a nominal trajectory $V = \{v_0, \ldots, v_{N-1}\}$, $Z = \{z_0, \ldots, z_N\}$

$\Rightarrow$ Complexity similar to nominal MPC.

Error dynamics - reachable and invariant sets

- Consider the error $e_i = x_i - z_i$ with feedback $u_i = v_i + K(x_i - z_i)$.
  The error dynamics are given by

$$\begin{aligned} e_{i+1} &= x_{i+1} - z_{i+1} \\ &= Ax_i + Bu_i + w_i - (Az_i + Bv_i) \\ &= \underbrace{(A + BK)}_{A_K} e_i + w_i. \end{aligned}$$

- A_K Schur stable and $\mathcal{W}$ compact

- Disturbance reachable set (DRS): $\mathcal{F}_{i+1} = A_K \mathcal{F}_i \oplus \mathcal{W}$, $\mathcal{F}_0 = \{0\}$,
  $\Rightarrow e_i \in \mathcal{F}_i = \bigoplus_{j=0}^{i-1} A_K^j \mathcal{W}$, if $e_0 = 0$.

- Robust positive invariant (RPI) set $\mathcal{E}$, such that $e_i \in \mathcal{E} \Rightarrow e_{i+1} \in \mathcal{E}$.
  For any $e_0 \in \mathcal{E}$, we have $e_k \in \mathcal{E}$.

- Note that $\mathcal{F}_i \subseteq \mathcal{F}_\infty \subseteq \mathcal{E}$ by definition.

Linear robust MPC

We describe two robust MPC approaches:

Constraint-tightening MPC [1]

Disturbance reachable sets (DRS) $\mathcal{F}_k$
$\Rightarrow$ Nominal MPC with tightened constraints and initial state $z_0 = x(k)$.

Tube-MPC [2]

Robust positive invariant (RPI) set $\mathcal{E}$
$\Rightarrow$ Nominal MPC with tightened constraints and initial state $z_0 \in x(k) \ominus \mathcal{E}$.

Both conceptual approaches have benefits and will be used throughout the lectures.

Outline

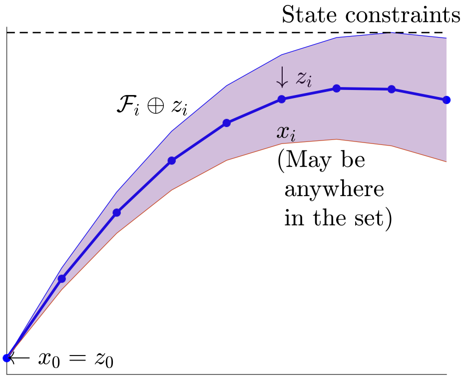
1. Linear robust MPC

   Error dynamics, reachable and invariant sets

   Constraint-tightening MPC

   Tube-MPC

Constraint-tightening MPC [1] - Illustration



Plan the nominal trajectory such that all possible state trajectories satisfy the **state constraint**

Constraint-tightening MPC - [1]

$$
\begin{aligned}
V_N^\star(x) = \min_{V} \quad & \sum_{i=0}^{N-1} l(z_i, v_i) + l_\mathrm{f}(z_N) \\
\text{s.t.} \quad & z_{i+1} = A z_i + B v_i, \quad i \in [0, N-1], \\
& z_i \in \overline{\mathcal{X}}_i = \mathcal{X} \ominus \mathcal{F}_i, \quad i \in [0, N-1], \\
& v_i \in \overline{\mathcal{U}}_i = \mathcal{U} \ominus K\mathcal{F}_i, \quad i \in [0, N-1], \\
& z_N \in \mathcal{X}_\mathrm{f} \ominus \mathcal{F}_N, \\
& z_0 = x
\end{aligned}
$$

- Nominal state and input predictions $Z = (z_0, \dots, z_N)$, $V = (v_0, \dots, v_{N-1})$
- Tightened nominal constraints $\overline{\mathcal{X}}_i$, $\overline{\mathcal{U}}_i$
- DRS $\mathcal{F}_i$ computed offline

- Initial state constraint: $z_0 = x$
- Control law: $\mu_{\mathrm{tightened}}(x) = v_0^\star(x)$
- Terminal set $\mathcal{X}_\mathrm{f}$ = (maximal) RPI set

Constraint-tightening MPC - theoretical properties

Theorem: Robust Stability of linear constraint-tightening MPC

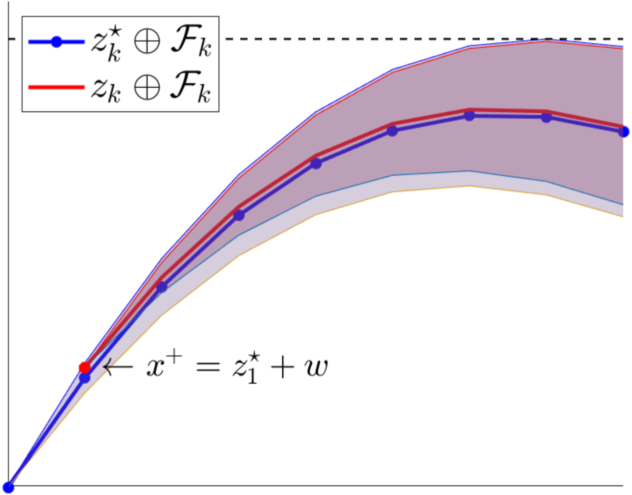
Suppose the terminal ingredients $(l_\mathrm{f}, \mathcal{X}_\mathrm{f}, \pi_\mathrm{f})$ are designed properly[a], the cost functions l, l_f are uniformly continuous, and the constraint-tightening MPC problem is feasible at $k = 0$ with initial condition $x(0)$.
Then, the constraint-tightening MPC is recursively feasible, $(x(k), u(k)) \in \mathcal{X} \times \mathcal{U}$, $\forall k \geq 0$, and the closed-loop system $x(k+1) = Ax(k) + B\mu_{\mathrm{tightened}}(x(k)) + w(k)$ is ISS on the feasible set $\mathcal{X}_N$ w.r.t. $w(k)$.

[a] For all $z \in \mathcal{X}_\mathrm{f}$, $w \in \mathcal{W}$: $l_\mathrm{f}(f(z, \pi_\mathrm{f}(z))) - l_\mathrm{f}(z) \leq -l(z, \pi_\mathrm{f}(z))$, $(z, \pi_\mathrm{f}(z)) \in \mathcal{X} \times \mathcal{U}$, $f(z, \pi_\mathrm{f}(z)) + w \in \mathcal{X}_\mathrm{f}$ with $\pi_\mathrm{f}(x) = Kx$.

Proof sketch:

- Let $(\{(v_0^\star, \ldots, v_{N-1}^\star\}, \{z_0^\star, \ldots, z_N^\star\})$ be the optimal solution at time k.
- New measured state $x(k+1) = Ax(k) + Bu(k) + w(k) = Az_0^\star + Bv_0^\star + w(k) = z_1^\star + w(k)$.

Constraint-tightening MPC - theoretical properties



- DRS around nominal trajectory
- Nominal candidate perturbed due to disturbance
- DRS around candidate prediction contained in previous DRS.

Constraint-tightening MPC - theoretical properties

Proof - cont.

- Consider the candidate solution based on the error feedback and terminal controller:
  $v_i = v^\star_{i+1} + K(z_i - z^\star_{i+1})$, $i = 0, \dots, N-1$, $\quad v^\star_N = Kz^\star_N$, $z^\star_{N+1} = A_K z^\star_N$.

- Recursive application yields:
  $z_i = z^\star_{i+1} + A_K^i w(k)$, $i = 0, \dots, N$.
  $v_i = v^\star_{i+1} + KA_K^i w(k)$, $i = 0, \dots, N-1$.

  Induction proof:
  $z_{i+1} = Az_i + Bv_i = A(z^\star_{i+1} + A_K^i w(k)) + B(v^\star_{i+1} + K(z_i - z^\star_{i+1})) = z^\star_{i+2} + A_K^{i+1} w(k)$.

- Nested prediction:
  $z_i \oplus \mathcal{F}_i = z^\star_{i+1} + A_K^i w(k) \oplus \mathcal{F}_i \subseteq z^\star_{i+1} \oplus A_K^i \mathcal{W} \oplus \mathcal{F}_i = z^\star_{i+1} \oplus \mathcal{F}_{i+1}$.

Constraint-tightening MPC - theoretical properties

Proof - cont.

- Nested prediction: $z_i \oplus \mathcal{F}_i \subseteq z^\star_{i+1} \oplus \mathcal{F}_{i+1}$.
- Terminal constraint:
  $z_N \oplus \mathcal{F}_N \subseteq z^\star_{N+1} \oplus \mathcal{F}_{N+1} = A_K(z^\star_N \oplus \mathcal{F}_N) \oplus \mathcal{W} \subseteq A_K \mathcal{X}_\mathrm{f} \oplus \mathcal{W} \overset{\text{RPI}}{\subseteq} \mathcal{X}_f$.
- Feasible candidate ensures recursive feasibility.
- Constraint satisfaction with $x = z^\star_0 \in \mathcal{X}$, $u = v^\star_0 \in \mathcal{U}$.
- ISS with ISS Lyapunov function $V^\star_N$ follows analogous to the proof in Lecture 2.[1] $\square$.

Robust constraint satisfaction & clearly defined region of attraction.
Only qualitative stability properties (ISS)

$\Rightarrow$ use tube-MPC for stronger stability properties

[1] Instead of continuity of $V^\star_N$, it suffices if the cost of the candidate sequence is continuous w.r.t. w.

Outline

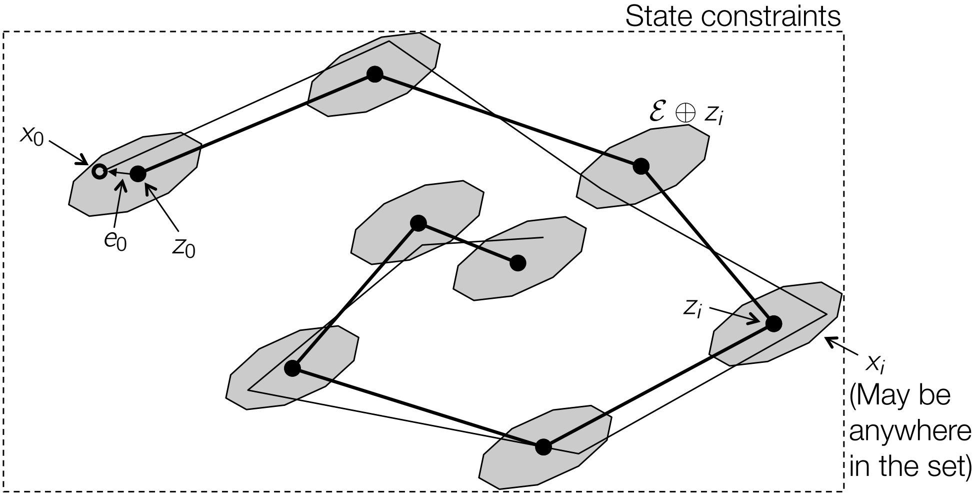
1. Linear robust MPC

    Error dynamics, reachable and invariant sets

    Constraint-tightening MPC

    Tube-MPC

Linear tube-MPC [2] - Illustration



Linear tube-MPC - [2]

$$
\begin{aligned}
V_N^\star(x) = \min_{V,\, z_0} \quad & \sum_{i=0}^{N-1} l(z_i, v_i) + l_{\mathrm{f}}(z_N) \\
\text{s.t.} \quad & z_{i+1} = A z_i + B v_i, \quad i \in [0, N-1], \\
& z_i \in \overline{\mathcal{X}} = \mathcal{X} \ominus \mathcal{E}, \quad i \in [0, N-1], \\
& v_i \in \overline{\mathcal{U}} = \mathcal{U} \ominus K\mathcal{E}, \quad i \in [0, N-1], \\
& z_N \in \mathcal{X}_{\mathrm{f}}, \\
& x \in \{z_0\} \oplus \mathcal{E}
\end{aligned}
$$

- Nominal state and input predictions $Z = (z_0, \ldots, z_N)$, $V = (v_0, \ldots, v_{N-1})$
- Tightened nominal constraints $\overline{\mathcal{X}}$, $\overline{\mathcal{U}}$
- RPI set $\mathcal{E}$

- Initial state constraint: $x \in z_0 \oplus \mathcal{E}$
- Control law: $\mu_{\mathrm{tube}}(x) = K(x - z_0^\star(x)) + v_0^\star(x)$

Linear tube-MPC - theoretical properties

Theorem: Robust Stability of linear tube-MPC

Suppose the terminal ingredients $(l_{\mathrm{f}}, \mathcal{X}_{\mathrm{f}}, \pi_{\mathrm{f}})$ are designed properly[a] and the tube-MPC problem is feasible at $k = 0$ with initial condition $x(0)$.
Then, the tube-MPC is recursively feasible, $(x(k), u(k)) \in \mathcal{X} \times \mathcal{U}$, $\forall k \geq 0$, and the state $x(k)$ converges to the set $\mathcal{E}$ for the closed-loop system $x(k+1) = Ax(k) + B\mu_{\mathrm{tube}}(x(k)) + w(k)$.

[a] Assumptions from Lecture 2 for nominal system with tightened constraints:
For all $z \in \mathcal{X}_{\mathrm{f}}$: $l_{\mathrm{f}}(f(z, \pi_{\mathrm{f}}(z))) - l_{\mathrm{f}}(z) \leq -l(z, \pi_{\mathrm{f}}(z))$, $(z, \pi_{\mathrm{f}}(z)) \in \overline{\mathcal{X}} \times \overline{\mathcal{U}}$, $f(z, \pi_{\mathrm{f}}(z)) \in \mathcal{X}_{\mathrm{f}}$.

Proof sketch:

- Let $(\{(v_0^\star, \ldots, v_{N-1}^\star\}, \{z_0^\star, \ldots, z_N^\star\})$ be the optimal solution at time k.
- Consider candidate $\{v_1^\star, \ldots, v_{N-1}^\star, \pi_{\mathrm{f}}(z_N^\star)\}$, $\{z_1^\star, \ldots, z_N^\star, Az_N + B\pi_{\mathrm{f}}(z_N^\star)\}$
- This nominal candidate Z, V satisfies the nominal dynamics, state, input, and terminal set constraints analogous to the nominal MPC proof (cf. Lecture 2).

Linear tube-MPC - theoretical properties

Proof - cont.

- The initial state constraint holds using:
  $x(k+1) - z_1^\star = (A+BK)(x(k) - z_0^\star) + w(k) \in (A+BK)\mathcal{E} \oplus \mathcal{W} \overset{\text{RPI}}{\subseteq} \mathcal{E}$.
- Feasible candidate ensures recursive feasibility.
- Constraint satisfaction follows from tightened constraint sets:
  $x(k) \in z_0^\star \oplus \mathcal{E} \subseteq \overline{\mathcal{X}} \oplus \mathcal{E} \subseteq \mathcal{X}$.
- Value function (nominal cost) $V_N^\star$ decreases analogous to the nominal proof in Lecture 2.
  $\Rightarrow$ z converges to the origin.
- Initial state constraint ensures x converges to $\mathcal{E}$. $\square$

$\Rightarrow$ Strong convergence/stability guarantees, robust constraint satisfaction, and clearly defined region of attraction.

Linear robust MPC

Constraint-tightening MPC [1]

Disturbance reachable sets (DRS) $\mathcal{F}_i$
$\Rightarrow$ Nominal MPC with tightened constraints and initial state $z_0 = x(k)$.

Tube-MPC [2]

Robust positive invariant (RPI) set $\mathcal{E}$
$\Rightarrow$ Nominal MPC with tightened constraints and initial state $z_0 \in x(k) \ominus \mathcal{E}$.

Commonalities:

- Offline design linear feedback K, compute sets $\mathcal{F}_i/\mathcal{E}$ containing error; and tighten constraints.
- Online solve a nominal MPC with tightened constraints and possibly initial state constraints.

Subtle differences:

- Tube-MPC has a smaller feasible set and RPI set increases complexity.
- Tube-MPC has simpler proof with stronger stability properties and simpler terminal set.

Can we extend this design to nonlinear systems?

Outline

1. Linear robust MPC
2. Nonlinear systems and incremental stability
3. Nonlinear tube-MPC

Outline

2. Nonlinear systems and incremental stability

    Error dynamics for nonlinear systems

    Incremental stability

    RPI sets using incremental stability

    Offline design of incremental Lyapunov function

Error dynamics for nonlinear systems

Uncertain constrained system

$$x(k+1) = f(x(k), u(k)) + w(k), \quad (x(k), u(k)) \in \mathcal{Z} \subseteq \mathcal{X} \times \mathcal{U}, \quad w(k) \in \mathcal{W}.$$

Goal: Try to extend tube-MPC to nonlinear systems
$\Rightarrow$ Use again a nominal system (z, v) and try to find an RPI set $\mathcal{E}$ for the error $e = x - z$.

- Drive 'nominal' (disturbance free) system to the origin
  $z(k+1) = f(z(k), v(k))$.
- Ensure that the real trajectory "tracks" the nominal trajectory by using an additional feedback
  $u(k) = \kappa(x(k), z(k), v(k))$.

Error dynamics for nonlinear systems

The "error dynamics" are given by

$$\begin{aligned} e_{i+1} &= x_{i+1} - z_{i+1} \\ &= f(x_i, u_i) + w_i - f(z_i, v_i) \\ &= f(z_i + e_i, \kappa(z_i + e_i, z_i, v_i)) - f(z_i, v_i) + w_i \\ &=: g(e_i, z_i, v_i) + w_i \neq g(e_i) + w_i. \end{aligned}$$

$\Rightarrow$ Dynamics of the error also depend on nominal state and input z, v!

Example: $f(x, u) = x^2 + u, \quad \kappa(x, z, v) = v,$
$f(x, u) - f(z, v) = x^2 - z^2 \neq e^2 = (x - z)^2.$

$\Rightarrow$ Exact disturbance propagation is difficult for nonlinear systems.

$\Rightarrow$ Need a different tool to ensure that the true system stays close to the nominal system.

Outline

2. Nonlinear systems and incremental stability

   Error dynamics for nonlinear systems

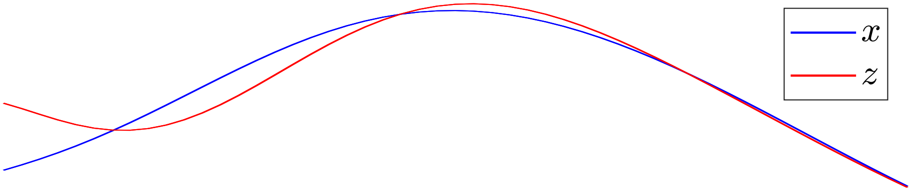
   Incremental stability

   RPI sets using incremental stability

   Offline design of incremental Lyapunov function

Incremental stability

- Characterize stability of trajectories $\Leftrightarrow$ incremental stability



- Incremental asymptotic stability:

$$\begin{aligned} \|x(k) - z(k)\| \leq& \beta(\|x_0 - z_0\|, k), \quad \beta \in \mathcal{KL}, \\ z(k+1) =& f(z(k), u(k)), \quad z(0) = z_0, \\ x(k+1) =& f(x(k), u(k)), \quad x(0) = x_0. \end{aligned}$$

Incremental ISS and stability of the error dynamics

- Incremental Lyapunov function $V_\delta(x, z)$ with exponential decay:

$$\begin{aligned} \alpha_1(\|x - z\|) \leq V_\delta(x, z) &\leq \alpha_2(\|x - z\|), \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty, \\ V_\delta(f(x, u), f(z, u)) &\leq \rho V_\delta(x, z), \quad \rho \in [0, 1). \end{aligned}$$

- Incremental/universal stabilizability: there exists a stabilizing feedback κ, such that

$$V_\delta(f(x, \kappa(x, z, v)), f(z, v)) \leq \rho V_\delta(x, z), \quad \rho \in [0, 1).$$

- Incremental input-to-state stability (i-ISS):

$$V_\delta(f(x, \kappa(x, z, v)) + w, f(z, v)) \leq \rho V_\delta(x, z) + \alpha_{\mathrm{w}}(\|w\|), \quad \rho \in [0, 1), \quad \alpha_{\mathrm{w}} \in \mathcal{K}.$$

Outline

2. Nonlinear systems and incremental stability

    Error dynamics for nonlinear systems

    Incremental stability

    RPI sets using incremental stability

    Offline design of incremental Lyapunov function

Robust positive invariant sets for nonlinear tube-MPC

- Incremental input-to-state stability (i-ISS):

$$V_\delta(f(x,\kappa(x,z,v))+w, f(z,v)) \leq \rho V_\delta(x,z) + \alpha_{\mathrm{w}}(\|w\|), \quad \rho \in [0,1), \quad \alpha_{\mathrm{w}} \in \mathcal{K}.$$

$\Rightarrow$ This property characterizes that the true uncertain system stays close to the nominal system if the disturbances are bounded and a feedback κ is applied

i-ISS: $\|x(k)-z(k)\| \leq \beta(\|x(0)-z(0)\|, k) + \gamma_{\mathrm{w}}(\max_{j\in[0,k-1]} \|w(j)\|)$, $\beta \in \mathcal{KL}$, $\gamma_{\mathrm{w}} \in \mathcal{K}$.

- Sublevel set of Lyapunov function $\Rightarrow$ Robust positive invariant (RPI) set:

$$\text{Parametrization: } \Omega := \{(x,z)|\ V_\delta(x,z) \leq \delta\}, \quad \delta > 0.$$

Robust positive invariant sets for nonlinear tube-MPC

Derivation:

- RPI condition: $(f(x,\kappa(x,z,v)+w), f(z,v)) \in \Omega$, $\forall w \in \mathcal{W}$, $(x,z) \in \Omega$, with $\Omega := \{(x,z) |\ V_\delta(x,z) \leq \delta\}$.
- Incremental input-to-state stability (i-ISS):

$$V_\delta(f(x,\kappa(x,z,v)) + w, f(z,v)) \leq \rho V_\delta(x,z) + \alpha_{\mathrm{w}}(\|w\|), \quad \rho \in [0,1), \quad \alpha_{\mathrm{w}} \in \mathcal{K}.$$

- Succesor state: $x^+ = f(x,\kappa(x,z,v)) + w$, $z^+ = f(z,v)$, disturbance bound $\overline{w} := \max_{w \in \mathcal{W}} \alpha_{\mathrm{w}}(\|w\|)$:

$$V_\delta(x^+, z^+) \leq \rho V_\delta(x,z) + \alpha_{\mathrm{w}}(\|w\|) \leq \rho\delta + \overline{w} \overset{!}{\leq} \delta.$$

$\Rightarrow$ Hold with equality for $\delta = \dfrac{\overline{w}}{1-\rho}$. $\square$

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2. Nonlinear systems and incremental stability

   Error dynamics for nonlinear systems

   Incremental stability

   RPI sets using incremental stability

   Offline design of incremental Lyapunov function

Offline design of incremental Lyapunov function

How to compute/design an incremental Lyapunov function?

- Simplest case – quadratic incremental Lyapunov function with linear feedback [3]:
  $V_\delta(x,z) = \|x-z\|_M,\ M \succ 0, \quad \kappa(x,z,v) = v + K(x-z)$.

- Define differential dynamics/linearization:

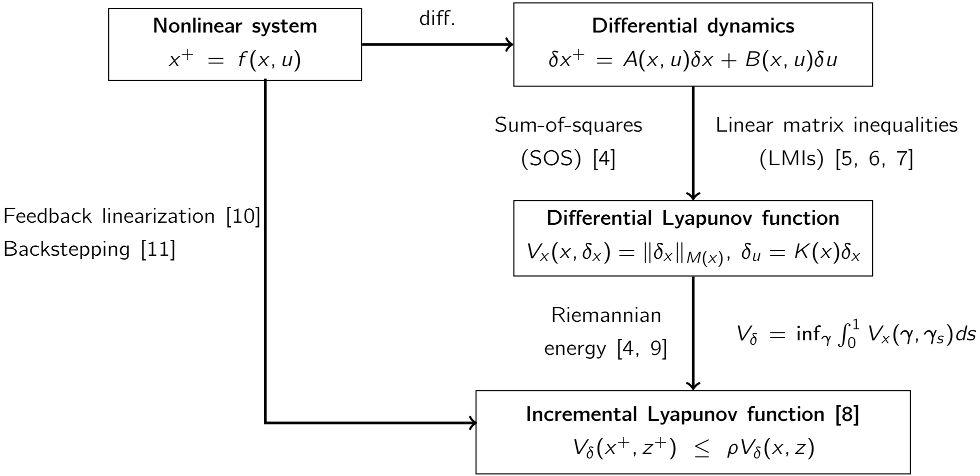
$$\delta_x(k+1) = \underbrace{[A(x,u) + B(x,u)K]}_{=:A_K(x,u)} \delta_x(k),$$

$$A(x,u) = \left.\left[\frac{\partial f}{\partial x}\right]\right|_{(x,u)}, \quad B(x,u) = \left.\left[\frac{\partial f}{\partial u}\right]\right|_{(x,u)}.$$

- Gradient theorem: $A_K^\top(x,u) M A_K(x,u) \preceq \rho^2 M,\ \forall (x,u)$
  $\Rightarrow$ δ-ISS condition: $\|f(x, v + K(x-z)) - f(z,v) + w\|_M \leq \rho \|x-z\|_M + \|w\|_M$.

- Matrices M, K can be designed using LMIs, cf. recitation.

Incremental Lyapunov function - General design



Outline

1. Linear robust MPC

2. Nonlinear systems and incremental stability

3. Nonlinear tube-MPC

Outline

Nonlinear tube-MPC

Uncertain constrained nonlinear system

$$x(k+1) = f(x(k), u(k)) + w(k), \quad (x(k), u(k)) \in \mathcal{Z} \subseteq \mathcal{X} \times \mathcal{U}, \quad w(k) \in \mathcal{W}.$$

Control goals

Closed loop should:

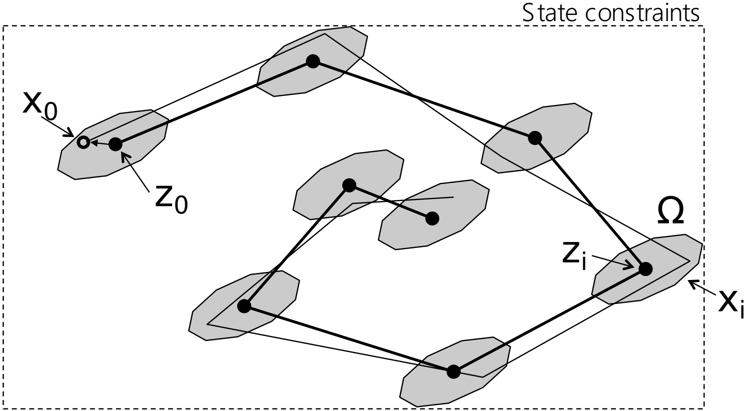
- satisfy constraints $(x(k), u(k)) \in \mathcal{Z}$ for all disturbance realizations,
- ensure stability (converge to a neighbourhood of the origin),
- for a large set of initial conditions.

Tube based robust MPC - [12]

$$
\begin{aligned}
V_N^\star(x) = \min_{V,\, z_0} \quad & \sum_{i=0}^{N-1} l(z_i, v_i) + l_{\mathrm{f}}(z_N) \\
\text{s.t.} \quad & z_{i+1} = f(z_i, v_i), \quad i \in [0, N-1], \\
& (z_i, v_i) \in \overline{\mathcal{Z}}, \quad i \in [0, N-1], \\
& z_N \in \mathcal{X}_{\mathrm{f}}, \\
& (x, z_0) \in \Omega
\end{aligned}
$$

- Nominal state and input predictions $Z = (z_0, \dots, z_N)$, $V = (v_0, \dots, v_{N-1})$
- Tightened nominal constraints $\overline{\mathcal{Z}} \subseteq \mathcal{Z}$
- initial state constraint: $(x, z_0) \in \Omega$
- Control law: $\mu_{\mathrm{tube}}(x) = \kappa(x, z_0^\star(x), v_0^\star(x))$

Tube-MPC - Illustration



Design of constraint tightening

- Constraints characterized by $\mathcal{Z} = \{(x,u) | \ g_j(x,u) \leq 0, \ j \in [1,p]\}$.
  - (Lipschitz) continuity of constraints: $g_j(x, \kappa(x,z,v)) - g_j(z,v) \leq c_j V_\delta(x,z)$, $c_j \geq 0$.
  - Constants c_j simple to compute, see example in recitation and [9, Prop. 5].

- Tightened constraint set:
  - $\overline{\mathcal{Z}} = \{(z,v) | \ g_j(z,v) + c_j\delta \leq 0, \ j \in [1,p]\}$.
  - Simple to compute; same complexity as original constraint set $\mathcal{Z}$.
  - Back-off $c_j\delta$ proportional to maximal disturbance.

- Tightened nominal constraints ensure $(x,u) \in \mathcal{Z}$ for closed-loop system $(z,v) \in \overline{\mathcal{Z}}$, $(x,z) \in \Omega = \{(x,z) | \ V_\delta(x,z) \leq \delta\}$ and feedback $u = \kappa(x,z,v)$.

  Derivation: $g_j(x,u) \leq g_j(z,v) + c_j V_\delta(x,z) \leq g_j(z,v) + c_j\delta \leq 0$

Outline

3. Nonlinear tube-MPC

   Setup and robust MPC formulation

   Theoretical analysis

   Overall algorithm & discussion

Offline design requirements/properties

- Robust positive invariant (RPI) set Ω (computed using incremental Lyapunov function V_δ):
  $(f(x, \kappa(x, z, v) + w), f(z, v)) \in \Omega$ for all $(x, z) \in \Omega$, $w \in \mathcal{W}$.

- Tightened constraint set $\overline{\mathcal{Z}}$:
  $(x, \kappa(x, z, v)) \in \mathcal{Z}$ for all $(z, v) \in \overline{\mathcal{Z}}$, $(x, z) \in \Omega$.

- Terminal set: For all $z \in \mathcal{X}_\mathrm{f}$, we have:
  - Positive invariance: $f(z, \pi_\mathrm{f}(z)) \in \mathcal{X}_\mathrm{f}$,
  - Constraint satisfaction: $(z, \pi_\mathrm{f}(z)) \in \overline{\mathcal{Z}}$,
  - Local Lyapunov function: $l_\mathrm{f}(f(z, \pi_\mathrm{f}(z))) - l_\mathrm{f}(z) \leq -l(z, \pi_\mathrm{f}(z))$.

- Positive definite and bounded cost:
  - $l(x, u) \geq \alpha_l(\|x\|)$, $\forall (x, u) \in \mathcal{X} \times \mathcal{U}$, $\alpha_l \in \mathcal{K}_\infty$,
  - $0 \in \mathrm{int}(\mathcal{X}_\mathrm{f})$ and $l_\mathrm{f}(x) \leq \alpha_\mathrm{f}(\|x\|)$, $\forall x \in \mathcal{X}_\mathrm{f}$, $\alpha_\mathrm{f} \in \mathcal{K}_\infty$.

Tube-MPC - Theoretical analysis

Theorem: Robust invariance and robust stability of tube-MPC

Let assumptions hold and assume that the nonlinear tube-MPC problem is feasible at $k = 0$ with initial condition $x(0)$. Then, the nonlinear tube-MPC problem is recursively feasible and the constraints are satisfied for all $k \geq 0$, for the closed-loop system $x(k+1) = f(x(k), \mu_{\mathrm{tube}}(x(k)) + w(k)$. Furthermore, the closed loop converges to the RPI set $\Omega_x = \{x|\ (x, 0) \in \Omega\}$.

Tube-MPC - Theoretical analysis

Proof: Let $Z^\star = \{z_0^\star, \ldots, z_N^\star\}$, $V^\star = \{v_0^\star, \ldots, v_{N-1}^\star\}$ be the optimal solution for $x(k)$.
As in the nominal MPC (Lecture 2), the sequence

$$Z = \{z_1^\star, \ldots z_N^\star, f(z_N^\star, \pi_{\mathrm{f}}(z_N^\star))\}, \quad V = \{v_1^\star, \ldots, v_{N-1}^\star, \pi_{\mathrm{f}}(z_N^\star)\}$$

satisfies the nominal dynamics, state, input, and terminal constraint.

At time $k+1$, we have

$$(x(k+1), z_1^\star) = (f(x(k), \kappa(x(k), v_0^\star, z_0^\star) + w(k)), f(z_0^\star, v_0^\star)) \in \Omega,$$

using $w(k) \in \mathcal{W}$, $(x(k), z_0^\star) \in \Omega$ and the fact that Ω is an RPI set.

$\Rightarrow$ Candidate $(V, z_1^\star)$ satisfies all constraints.

$\Rightarrow$ tube-MPC problem is robustly recursively feasible for all $k \geq 0$ (using induction).

Tube-MPC - Theoretical analysis

Proof - cont.

- Constraint satisfaction of closed-loop system due to tightened constraints

$$(z_0^\star, v_0^\star) \in \overline{\mathcal{Z}},\ (x(k), z_0^\star) \in \Omega \quad \Rightarrow \quad (x(k), u(k)) = (x(k), \kappa(x(k), z_0^\star, v_0^\star)) \in \mathcal{Z}.$$

- "Stability" of the nominal state z follows the same arguments as nominal MPC:

$$V_N^\star(x(k+1)) \leq V_N(V, z_1^\star) \leq V_N^\star(x(k)) - l(z_0^\star(x(k)), v_0^\star(x(k))) \leq V_N^\star(x(k)) - \alpha_l(\|z_0^\star(x(k))\|).$$

  Telescopic sum: $\sum_{k=0}^{T-1} \alpha_l(\|z_0^\star(x(k))\|) \leq V_N^\star(x(0)) - V_N^\star(x(T)) < \infty \Rightarrow \lim_{k\to\infty} z_0^\star(x(k)) = 0.$

  The initial state constraint ensures $(x(k), z_0^\star(x(k))) \in \Omega \Rightarrow \lim_{k\to\infty} x(k) \in \Omega_x.$ $\square$

For $(z_0^\star, v_0^\star) = 0$, MPC control law reduces to the stabilizing feedback $\kappa(x, 0, 0)$.

Outline

3. Nonlinear tube-MPC

    Setup and robust MPC formulation

    Theoretical analysis

    Overall algorithm & discussion

Overall algorithm - Tube-MPC

— Offline —

1. Design a stabilizing feedback κ with an RPI set Ω.
   (E.g. using incremental stability, linearization and LMIs)
2. Compute tightened constraints $\overline{\mathcal{Z}}$.
3. Design terminal cost l_f and terminal set $\mathcal{X}_\mathrm{f}$ for nominal system.

— Online—

1. Measure/estimate state x.
2. Solve tube-MPC problem.
3. Apply $u = \kappa(x, z_0^\star(x), v_0^\star(x))$.

Tube-MPC - Discussion

Pro:

- Same (simple) conceptual idea and algorithm as linear tube-MPC
- Strong guarantees for closed loop despite disturbances (robust by design):
  - robust constraint satisfaction,
  - robust stability/performance,
  - feasible set = region of attraction.

Limitation:

- Computing Ω, κ for general nonlinear systems can be difficult.
- Robust MPC can be conservative (small feasible set, system is often not operated at the limits, $w \in \mathcal{W}$ is a crude/conservative uncertainty characterization).

Robust MPC for nonlinear uncertain systems - summary

Idea:

- Compensate for noise in nominal prediction using a stabilizing feedback κ
- Method: incremental stability & tube-formulation

Benefits:

- Specify robustness using disturbance bound $\mathcal{W}$
- Guaranteed stability and safety properties
- Feasible set is invariant - we know exactly when the controller will work

Cons/Limitations:

- Often conservative
- Offline design challenging for nonlinear systems

$\Rightarrow$ Next Lecture: reducing conservatism.

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