AMPyC

Chapter 2: Robustness of Nominal MPC

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Ideal Optimal Control Problem

Optimal control problem that we ideally would like to solve – the robust case

$$
\begin{aligned}
\min_{\{\pi_k\}} \quad & \max_{W,\theta} \left(\sum_{k=0}^{\bar{N}} l_k(x(k), u(k)) \right) \\
\text{s.t.} \quad & x(k+1) = f(x(k), u(k), w(k); \theta), \\
& u(k) = \pi_k(x(0), \dots, x(k), u(0) \dots, u(k-1)), \\
& X \in \mathcal{X}^{\bar{N}}, U \in \mathcal{U}^{\bar{N}} \; \forall W \in \mathcal{W}^{\bar{N}}, \theta \in \Theta, \\
& x(0) = x_{\text{init}}
\end{aligned}
$$

- State sequence $X = [x(0)^\top, \dots, x(\bar{N}-1)^\top]^\top$
- Input sequence $U = [u(0)^\top, \dots, u(\bar{N}-1)^\top]^\top$
- Disturbance sequence $W = [w(0)^\top, \dots, w(\bar{N}-1)^\top]^\top$
- Objective function $L(X, U)$
- Constraints $\mathcal{X}$, $\mathcal{U}$
- Uncertainties $\theta \in \Theta$, $W \in \mathcal{W}^{\bar{N}}$,

MPC under Uncertainty – Robust Case

$$
\begin{aligned}
\min_{\{\pi_k\}} \quad \max_{W,\theta} \quad & \left(\sum_{k=0}^{\bar{N}} l_k(x(k), u(k)) \right) \\
\text{s.t.} \quad & x(k+1) = f(x(k), u(k), w(k); \theta), \\
& u(k) = \pi_k(\cdot), \\
& w(k) \in \mathcal{W},\ \theta \in \Theta, \\
& x(k) \in \mathcal{X}, \\
& u(k) \in \mathcal{U}, \\
& x(0) = x_{\text{init}}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min_{\{\pi_i\}} \quad \max_{W,\theta} \quad & \left(l_f(x_N) + \sum_{i=0}^{N-1} l_i(x_i, u_i) \right) \\
\text{s.t.} \quad & x_{i+1} = f(x_i, u_i, w_i; \theta), \\
& u_i = \pi_i(\cdot), \\
& w_i \in \mathcal{W},\ \theta \in \Theta, \\
& x_i \in \mathcal{X}, \\
& u_i \in \mathcal{U}, \\
& x_N \in \mathcal{X}_f, \\
& x_0 = x(k)
\end{aligned}
$$

- Solve over shortened horizon
- Restrict policy class (open-loop sequence, state feedback,...)

Certainty Equivalent Model Predictive Control

$$
\begin{aligned}
\min_{X,U} \quad & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{subj. to} \quad & x_{i+1} = f(x_i, u_i, \hat{w}_i, \hat{\theta}),\ \ i = 0, \dots, N-1 \\
& x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \dots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

- $\hat{w}_0, \dots, \hat{w}_{N-1}$ are predicted values of the disturbance $w(k), \dots, w(k+N-1)$ based on $x(0), \dots, x(k)$, $\hat{\theta}$ is parameter estimate

$\rightarrow$ Deterministic constrained optimization problem

- widely used and can work well
- based on (potentially bad) approximation: future disturbance values are exactly as predicted/estimated, there is no future uncertainty

Nominal MPC Problem – Ignore The Noise

$$
\begin{aligned}
\min_{X,U} \quad & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{subj. to} \quad & \\
& x_{i+1} = f(x_i, u_i),\ \ i = 0, \ldots, N-1 \\
& x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

- Predicted state sequence $X = [x_0^\top, \ldots, x_N^\top]^\top$
- Predicted input sequence $U = [u_0^\top, \ldots, u_N^\top]^\top$

$\rightarrow$ Deterministic constrained optimization problem

Recall: Nominal MPC - Main goals

Nominal system
$x(k+1) = f(x(k), u(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U}$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satifies constraints : $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$ for all k
2. Is stable: $\lim_{k\to\infty} x(k) = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x(0) \mid \text{Conditions 1-3 are met}\}$

MPC for Bounded Uncertainties - Robust setting

Uncertain constrained system
$x(k+1) = f(x(k), u(k), w(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$ for all k and for all disturbance realizations
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

MPC for Bounded Uncertainties - Robust setting

Uncertain constrained system
$x(k+1) = f(x(k), u(k), w(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$

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3. Optimizes (expected/worst-case) “performance”
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Question 1:
What happens if we ignore the noise and apply the nominal MPC control law?
Can we achieve these properties with nominal MPC?

Contents – 2: Nominal MPC

- Recall and refine Lyapunov stability theory
- Nominal MPC theory
  - Prove recursive feasibility
  - Prove asymptotic stability
- Understand nominal MPC properties for uncertain systems / disturbances
  - Learn concept of input-to-state stability
  - Understand assumptions under which nominal MPC provides input-to-state stability
  - Investigate special cases where this is true

Outline

1. Nominal MPC theory (Recap and Refinement)

2. Robustness of Nominal MPC

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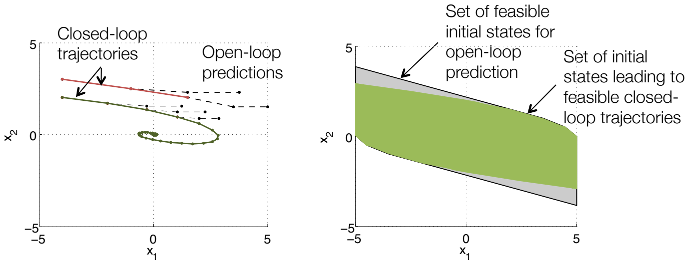
2. Robustness of Nominal MPC

Main Challenge in MPC Theory

Theoretical challenges in MPC result from the finite-horizon problem and receding horizon implementation: mismatch between prediction and closed-loop

Goal:

- Recursive feasibility
- Constraint satisfaction
- Stability / Convergence



Outline

1. Nominal MPC theory (Recap and Refinement)

   Problem Setup and Definitions

   Lyapunov stability

   Recap MPC feasibility & stability

System definition

Consider the discrete-time time-invariant nonlinear system

$$x(k+1) = f(x(k), u(k), w(k)), \;\; k \geq 0 \tag{1}$$

- Consider the initial state $x(0)$, a sequence of control inputs U and disturbances W. The solution of system (1) at sampling time k is denoted as

$$\phi(k, x(0), U, W),$$

where $\phi(0, x(0), U, W) = x(0)$.
- The nominal model of the plant (1) denotes the system considering zero disturbance and is given by

$$\bar{x}(k+1) = \bar{f}(\bar{x}(k), u(k)), \;\; k \geq 0, \tag{2}$$

where $\bar{f}(x, u) = f(x, u, 0)$.

The solution to (2) for a given initial state x(0) is $\bar{\phi}(k, x(0), U) := \phi(k, x(0), U, 0)$.
- Consider that system (1) is controlled by control law $u(k) = \pi(x(k))$.
Then the closed-loop system is expressed as:

Assumptions and Requirements

Consider the discrete-time time-invariant nonlinear system

$$x(k+1) = f(x(k), u(k), w(k)),\ \ k \geq 0$$

- The system has an equilibrium point at the origin, i.e. $f(0,0,0) = 0$.
- The control input and state of the plant must fulfill the following constraints:

$$x(k) \in \mathcal{X}, u(k) \in \mathcal{U}$$

- The constraint sets $\mathcal{X}, \mathcal{U}$ contain the origin in their interior
- The disturbance w is such that $w(k) \in \mathcal{W}$ for all $k \geq 0$, where $\mathcal{W} \subset \mathbb{R}^{n_w}$ is a compact set containing the origin.

Outline

1. Nominal MPC theory (Recap and Refinement)

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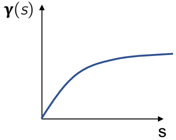
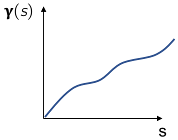
   Recap MPC feasibility & stability

Notation and basic definitions

$\mathcal{K}$-**function**: A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$, if it is continuous, strictly increasing and $\gamma(0) = 0$.



$\mathcal{K}_\infty$-**function**: A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$, if it is a $\mathcal{K}$-function and $\gamma(s) \to \infty$ as $s \to \infty$.



$\mathcal{KL}$-**function**: A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$, if

- for each fixed $k \geq 0$, $\beta(\cdot, k)$ is of class $\mathcal{K}$ and
- for each fixed $s \geq 0$, $\beta(s, \cdot)$ is non-increasing and $\beta(s, k) \to 0$ as $k \to \infty$.

Notation and basic definitions

Positive definite function: A function $V : \mathbb{R}^a \to \mathbb{R}_{\geq 0}$ is called positive definite, if $V(0) = 0$ and there exists a $\mathcal{K}$-function α such that $V(x) \geq \alpha(\|x\|)$

Uniform continuity A function $f(x, y) : A \times B \to \mathbb{R}^c$ with $A \subseteq \mathbb{R}^a$, $B \subseteq \mathbb{R}^b$, is uniformly continuous in x for all $x \in A$ and $y \in B$, iff there exists a $\mathcal{K}_\infty$-function σ such that

$$\|f(x_1, y) - f(x_2, y)\| \leq \sigma(\|x_1 - x_2\|) \quad \forall x_1, x_2 \in A, \forall y \in B.$$

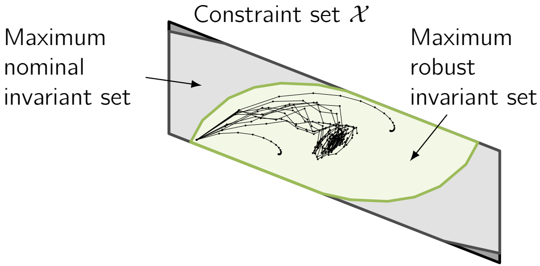
Recall: Robust Invariance

Robust constraint satisfaction, for **autonomous** system $x(k+1) = f(x(k), w(k))$, or **closed-loop** system $x(k+1) = f(x(k), \pi(x(k)), w(k))$ for a **given** controller π.

Robust Positive Invariant set

A set $\mathcal{O}^{\mathcal{W}}$ is said to be a robust positive invariant set for the autonomous system $x(k+1) = f(x(k), w(k))$ if

$$x \in \mathcal{O}^{\mathcal{W}} \quad \Rightarrow \quad f(x, w) \in \mathcal{O}^{\mathcal{W}} \quad , \quad \text{for all } w \in \mathcal{W}$$



If we have a (robust) invariant set $\mathcal{X}_f \subseteq \mathcal{X}$ and $\pi(\mathcal{X}_f) \subseteq \mathcal{U}$, then it provides a set of initial states from which the trajectory will never violate the system constraints if we apply the controller π.

Lyapunov stability

Show: Origin is an asymptotically stable equilibrium point for controlled system $x(k+1) = \bar{f}_\pi(x(k))$.
Assume: $\bar{f}_\pi(\cdot)$ is locally bounded.

Definition: Asymptotic stability

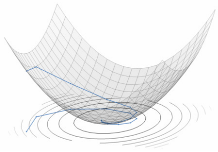
Let Ω be a positive invariant set for system $x(k+1) = \bar{f}_\pi(x(k))$. The origin is asymptotically stable in Ω, if there exists a $\mathcal{KL}$-function β such that for any $x(0) \in \Omega$:

$$\|\bar{\phi}_\pi(j, x(0))\| \leq \beta(\|x(0)\|, j) \quad \forall j \geq 0$$

Definition: Lyapunov function

Let Ω be a positive invariant set for system $x(k+1) = \bar{f}_\pi(x(k))$. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function in Ω for $x(k+1) = \bar{f}_\pi(x(k))$, if there exist three $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$ such that for any $x \in \Omega$:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$V(\bar{f}_\pi(x)) - V(x) \leq -\alpha_3(\|x\|)$$



Lyapunov stability

Lyapunov stability theorem

Let Ω be a positive invariant set for system $x(k+1) = \bar{f}_\pi(x(k))$. If there exists a Lyapunov function in Ω for system $x(k+1) = \bar{f}_\pi(x(k))$, then the origin is asymptotically stable in Ω.

If $\Omega = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Outline

1. Nominal MPC theory (Recap and Refinement)

   Problem Setup and Definitions

   Lyapunov stability

   Recap MPC feasibility & stability

Feasibility and stability in MPC - Main idea

Main idea: Select terminal cost and constraints to ensure feasibility and stability by design:

$$
\begin{aligned}
V_N^\star(x(k)) = \quad & \min_{U} && l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) && \text{Terminal Cost} \\
& \text{subj. to} && && \\
& && x_{i+1} = \bar{f}(x_i, u_i),\ \ i = 0, \dots, N-1 && \\
& && x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \dots, N-1 && \\
& && x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& && x_0 = x(k) &&
\end{aligned}
$$

- $l_f(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon
- $\mathcal{X}_N$ denotes the feasible set
- Let $V_N(x, U) = l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$ with $x_{i+1} = \bar{f}(x_i, u_i)$

$\rightarrow$ Applied receding horizon control law: $\pi_N^{\text{nom}}(x) = u_0^\star(x)$

Stability of nominal MPC - Assumptions

1. Stage cost is positive definite, i.e. there exists a $\mathcal{K}_\infty$ function $\alpha_l(\cdot)$ such that

$$l(x,u) \geq \alpha_l(\|x\|) \quad \forall x \in \mathcal{X}, u \in \mathcal{U}$$

2. Terminal set is **positive invariant** under the local control law $\pi_f(x)$:

$$x_{i+1} = \bar{f}(x_i, \pi_f(x_i)) \in \mathcal{X}_f, \;\; \text{for all } x_i \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$, origin is contained in the interior of $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\; \pi_f(x) \in \mathcal{U}, \;\; \text{for all } x \in \mathcal{X}_f; \quad 0 \in \text{int}(\mathcal{X}_f)$$

3. Terminal cost is a **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$l_f(x_{i+1}) - l_f(x_i) \leq -l(x_i, \pi_f(x_i)), \;\; \text{for all } x_i \in \mathcal{X}_f,$$

   and there exist $\mathcal{K}_\infty$ functions $\alpha_{1,f}(\cdot), \alpha_{2,f}(\cdot)$ such that

$$\alpha_{1,f}(\|x\|) \leq l_f(x) \leq \alpha_{2,f}(\|x\|), \quad \text{for all } x \in \mathcal{X}_f$$

Stability of nominal MPC

Under those 3 assumptions:

Theorem

The feasible set $\mathcal{X}_N$ is positive invariant and the origin is asymptotically stable for the closed-loop system under the MPC control law $\pi_N^{\text{nom}}(x)$, i.e.

$$x(k+1) = \bar{f}(x(k), \pi_N^{\text{nom}}(x(k))).$$

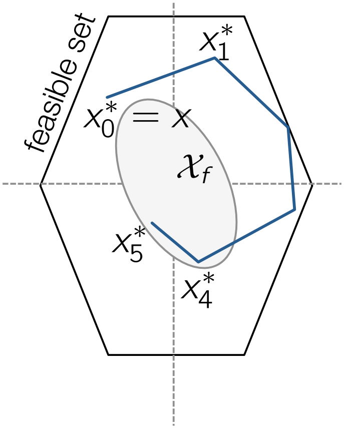
with region of attraction $\mathcal{X}_N$.

Two main steps to prove this result:

1. Prove recursive feasibility
2. Prove asymptotic stability

Recursive feasibility of MPC - Outline of the proof

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \dots,\ x_N^\star\}$ the corresponding state trajectory



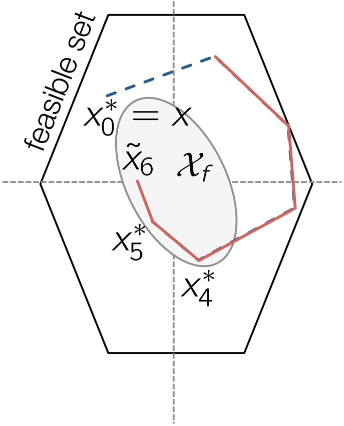
Recursive feasibility of MPC - Outline of the proof

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \ldots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \ldots,\ x_N^\star\}$ the corresponding state trajectory

- At $x(k+1) = x_1^\star$, the control sequence $\tilde{U} = \{u_1^\star,\ u_2^\star,\ \ldots,\ \pi_f(x_N^\star)\}$ is feasible:

  $x_N^\star$ is in $\mathcal{X}_f \rightarrow \pi_f(x_N^\star)$ is feasible

  and $\bar{f}(x_N^\star, \pi_f(x_N^\star))$ in $\mathcal{X}_f$



$\Rightarrow$ **Terminal constraint provides recursive feasibility**

$\Rightarrow$ **Feasible set $\mathcal{X}_N$ is invariant.**

Asymptotic stability of MPC - Outline of the proof

Idea: Prove that optimal MPC cost $V_N^\star(x) = \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star) + l_f(x_N^\star)$ is a Lyapunov function

(where $x_0^\star, \ldots, x_N^\star, u_0^\star, \ldots u_{N-1}^\star$ is the optimizer of the nominal MPC problem at x)

- Descent property $V_N^\star(\bar{f}(x, \pi_N^{\text{nom}}(x))) - V_N^\star(x) = V_N^\star(x^+) - V_N^\star(x) \leq -\alpha_3(\|x\|)$

  Idea: Show descent of cost of suboptimal shifted sequence

  At $x^+ = x_1^\star,\ \tilde{U} = \{u_1^\star,\ u_2^\star,\ \ldots,\ \pi_f(x_N^\star)\}$ is feasible & sub-optimal

$$
\begin{aligned}
V_N(x^+, \tilde{U}) &= \sum_{i=1}^{N-1} l(x_i^\star, u_i^\star) + l(x_N^\star, \pi_f(x_N^\star)) + l_f(\bar{f}(x_N^\star, \pi_f(x_N^\star))) \\
&= \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star) - l(x_0^\star, u_0^\star) + l_f(x_N^\star) - l_f(x_N^\star) + l(x_N^\star, \pi_f(x_N^\star)) + l_f(\bar{f}(x_N^\star, \pi_f(x_N^\star))) \\
&= V_N^\star(x) - l(x, u_0^\star) + \underbrace{l_f(\bar{f}(x_N^\star, \pi_f(x_N^\star))) - l_f(x_N^\star) + l(x_N^\star, \pi_f(x_N^\star))}_{\leq 0 \text{ by Assumption 3}}
\end{aligned}
$$

$$
\implies V_N^\star(x^+) - V_N^\star(x) \leq V_N(x^+, \tilde{U}) - V_N^\star(x) \leq -l(x, u_0^\star) \leq -\alpha_l(\|x\|) \quad \forall x \in \mathcal{X}_N
$$

Asymptotic stability of MPC - Outline of the proof

- Lower bound: $\alpha_1(\|x\|) \leq V_N^\star(x)$

$$V_N^\star(x) \geq l(x, u_0^\star(x)) \geq \alpha_l(\|x\|) \quad \forall x \in \mathcal{X}_N$$

- $V_N^\star(x) \leq \alpha_2(\|x\|)$
  - Show $V_N^\star(x) \leq l_f(x)\ \forall x \in \mathcal{X}_f$
    $\rightarrow$ follows from monotonity of the value function: $V_{j+1}^\star(x) \leq V_j^\star(x)$ for all $x \in \mathcal{X}_j$ (see [1])
  - Since $l_f(x) \leq \alpha_{2,f}(\|x\|)$: $V_N^\star(x) \leq \alpha_{2,f}(\|x\|)\ \forall x \in \mathcal{X}_f$
  - Since the origin lies in the interior of $\mathcal{X}_f$, this bound can be extended to a similar bound in $\mathcal{X}_N$
    $\rightarrow$ There exists a $\mathcal{K}_\infty$ function α_2 such that $V_N^\star(x) \leq \alpha_2(\|x\|)\ \forall x \in \mathcal{X}_N$

$V_N^\star(x)$ **is a Lyapunov function**
$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable in $\mathcal{X}_N$

Note: Proof can be extended to weaker assumptions [1].

Outline

1. Nominal MPC theory (Recap and Refinement)

2. Robustness of Nominal MPC

MPC for bounded uncertainties - Robust setting

Uncertain constrained nonlinear system
$x(k+1) = f(x(k), u(k), w(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$

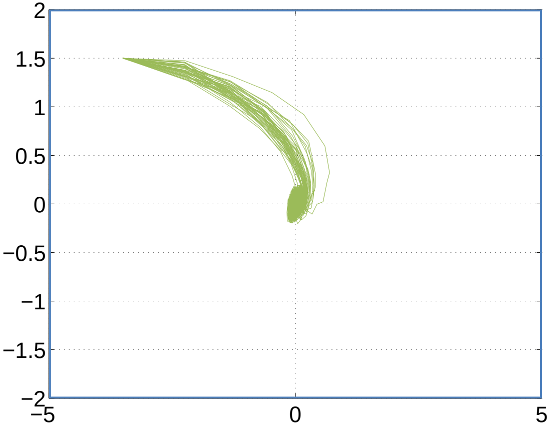
Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all disturbance realizations
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Question 1:
What happens if we ignore the noise and apply the nominal MPC control law?
Can we achieve these properties with nominal MPC?

Example

Consider double integrator system (linear) with additive disturbances, but we pretend it's not there in the controller.



- 100 trajectories with different noise realizations
- Seems to work fine?!

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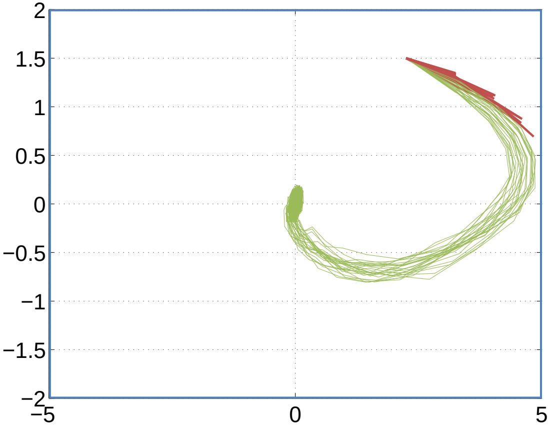


- 100 trajectories with different noise realizations
- Seems to work fine?!
- Can no longer be certain it will work!
- For some states it should work

Counterexample – When nominal MPC is non-robust

Consider the 2-dimensional discontinuous autonomous system

$$x(k+1) = f(x(k)),\ f(x) = \begin{cases} [0, \|x\|]^T & x_1 \neq 0 \\ [0, 0]^T & \text{otherwise} \end{cases}$$

Example initial state $x(0) = [1, 1]^T$:

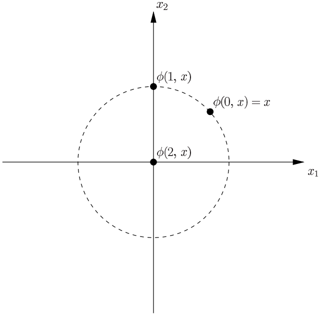
$$\phi(1; x(0)) = [0, \sqrt{2}]^T, \phi(2; x(0)) = [0, 0]^T$$

In fact, all solutions satisfy

$$\|\phi(k; x(0))\| \leq \beta(\|x(0)\|, k), \quad \text{with } \beta(\|x(0)\|, k) = 2\left(\frac{1}{2}\right)^k \|x(0)\|$$

where β is a $\mathcal{KL}$-function.

$\rightarrow$ Origin is globally asymptotically stable.



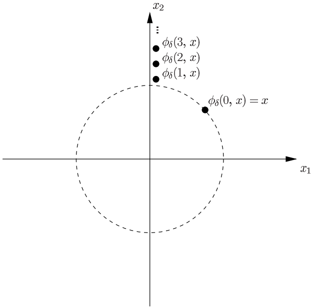
Source: [3]

Counterexample – When nominal MPC is non-robust

Consider now a perturbed system

$$x(k+1) = \begin{bmatrix} \delta \\ \|x(k)\| + \delta \end{bmatrix}$$

where $\delta > 0$ is a constant perturbation that causes x_1 to remain strictly positive.



Take $x(0) = \epsilon \cdot [1, 1]^T$:

$$x_1(k) = \delta \text{ for } k \geq 1,\ x_2(k) > \epsilon\sqrt{2} + k\delta \to \infty \text{ as } k \to \infty$$

no matter how small δ and ϵ are.

$\to$ Origin is unstable in the presence of an arbitrarily small perturbation.

Source: [3]

MPC for bounded uncertainties - Robust setting

Uncertain constrained nonlinear system
$$x(k+1) = f(x(k), u(k), w(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all k and for all disturbance realizations
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Question 1:
What happens if we ignore the noise and apply the nominal MPC control law?
Under which assumptions can we achieve these properties with nominal MPC?

$\rightarrow$ We need a notion of robust stability.

Outline

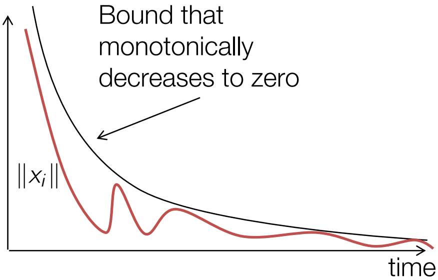
2. Robustness of Nominal MPC

    Input-to-state stability

    Input-to-state stability of nominal MPC

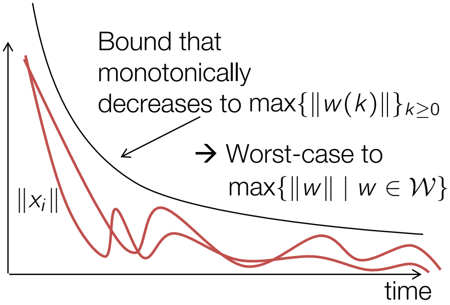
Input-to-state stability – Main idea

Asymptotic stability



System converges to zero

ISS stability



System converges to set around zero, whose size is determined by size of the noise

Input-to-state stability

Consider system

$$x(k+1) = f_\pi(x(k), w(k)),\ k \geq 0, \qquad w \in \mathcal{W} \tag{4}$$

Input-to-state Stability (ISS) in Γ

Suppose that $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for (4), including the origin as an interior point. System (4) is called input-to-state stable in Γ with respect to w, if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ_2 such that

$$\|\phi_\pi(k, x(0), W_{[k-1]})\| \leq \beta(\|x(0)\|, k) + \gamma_2(\|W_{[k-1]}\|)$$

for all $x(0) \in \Gamma$, $W_{[k-1]} \in \mathcal{W}^{k-1}$ and $k \geq 0$, where $\|W_{[k-1]}\| := \max_{0 \leq j \leq k-1}\{\|w(j)\|\}$.

Input-to-state stability – Lyapunov function

ISS Lyapunov function in Γ

Suppose that $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for (4), including the origin as an interior point. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function in Γ for system (4) with respect to w, if there exist suitable $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$, a $\mathcal{K}$-function λ_2 such that for all $x \in \Gamma, w \in \mathcal{W}$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$V(f_\pi(x, w)) - V(x) \leq -\alpha_3(\|x\|) + \lambda_2(\|w\|)$$

Theorem

Suppose that $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for (4), including the origin as an interior point. If system (4) admits an ISS-Lyapunov function in Γ w.r.t. w, then it is ISS in Γ w.r.t. w.

Remarks

- Input-to-state stability, as previously stated, holds for any given disturbance set.
  But:
  - The region of attraction (RPI set Γ) depends on the disturbance set.
  - The trajectories of the system converge asymptotically to a set which depends on the disturbance bound.

$\rightarrow$ Inherent robustness of nominal MPC corresponds to the existence of a small enough disturbance bound such that ISS (or another notion of robust asymptotic stability) holds.

- There are direct connections between ISS and other robust stability notions, such as the asymptotic gain, or robust stability margin (see [2,3] for more details).

Outline

2. Robustness of Nominal MPC

   Input-to-state stability

   Input-to-state stability of nominal MPC

Nominal MPC Controller

$$
\begin{aligned}
V_N^\star(x(k)) = \quad \min_U \quad & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{subj. to} \quad & \\
& x_{i+1} = \bar{f}(x_i, u_i), \ \ i = 0, \dots, N-1 \\
& x_i \in \mathcal{X}, \ \ u_i \in \mathcal{U}, \ \ i = 0, \dots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

- Assume that $\bar{f}(x, u)$ is uniformly continuous in x for all $x \in \mathcal{X}, u \in \mathcal{U}$
- Assume that $\mathcal{X}, \mathcal{U}$ contain the origin in their interior
- $\mathcal{X}_N$ denotes the feasible set
- Let $V_N(x, U) = l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$ with $x_{i+1} = \bar{f}(x_i, u_i)$

$\rightarrow$ Applied receding horizon control law: $\pi_N^{\text{nom}}(x) = u_0^\star(x)$

ISS of Nominal MPC – Main Result

Consider system

$$x(k+1) = f(x(k), u(k), w(k)),\ k \geq 0, \quad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W} \tag{5}$$

Consider the nominal MPC problem and let $\mathcal{X}_N$ be the feasible set and let $V_N^\star(x)$ be the corresponding optimal cost. Let the terminal cost and constraint satisfy the standard assumptions such that the closed-loop system under the resulting nominal MPC control law $\pi_N^{\mathrm{nom}}(x)$ is asymptotically stable and $V_N^\star(x)$ is a Lyapunov function in $\mathcal{X}_N$.

Let $f(x, \pi_N^{\mathrm{nom}}(x), w)$ be uniformly continuous in w for all $x \in \mathcal{X}_N$, $w \in \mathcal{W}$. Suppose one of the following conditions holds:

1. Function $\bar{f}(x, \pi_N^{\mathrm{nom}}(x))$ is uniformly continuous in x for all $x \in \mathcal{X}_N$,
2. The optimal cost $V_N^\star(x)$ is uniformly continuous in $\mathcal{X}_N$,

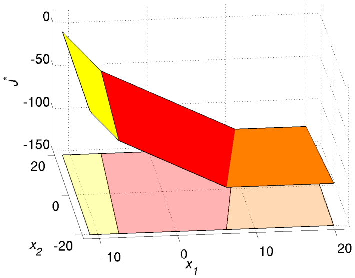
Then, there exists an RPI set $\Omega_r \subseteq \mathcal{X}_N$ for a sufficiently small bound on the disturbances w, such that the system (5) controlled by the nominal MPC controller $u(k) = \pi_N^{\mathrm{nom}}(x)$ is ISS in Ω_r.

Note: In Ω_r recursive feasibility is ensured despite disturbances.

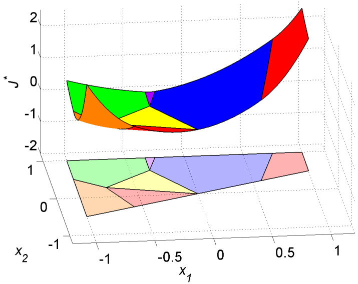
Special cases when nominal MPC leads to ISS – Case 1 [2]

C1: If the nominal model is linear, the cost function $V_N(x, U)$ is linear or quadratic, and the sets of constraints are polytopes, then the optimal cost $V_N^\star(x)$ is uniformly continuous in $\mathcal{X}_N$.

Recall: In this case the optimal cost function is a piece-wise linear/quadratic function[1]. Then $V_N^\star(x)$ is Lipschitz in $\mathcal{X}_N$ and hence uniformly continuous.



Example optimal cost for linear cost function



Example optimal cost for quadratic cost function

[1] Bemporad et al., "Model predictive control based on linear programming – the explicit solution". IEEE Trans. on Automatic Control, 47(12), 1974-1985, 2002.

Special cases when nominal MPC leads to ISS – Case 2 [2]

C2: If the plant has only constraints on the inputs, the terminal region is $\mathcal{X}_f = \mathbb{R}^n$ (i.e. $\mathcal{X}_N = \mathbb{R}^n$), and $V_N(x, U)$ is uniformly continuous in x for all $x \in \mathbb{R}^n$ and $U \in \mathcal{U}^N$, then the optimal cost $V_N^\star(x)$ is uniformly continuous in $\mathbb{R}^n$.

Note: If $f(x, u, w)$ is uniformly continuous in x, $l(x, u)$ is uniformly continuous in x and $V_f(x)$ is uniformly continuous for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, $w \in \mathcal{W}$, then uniform continuity of $V_N(x, U)$ holds.

Special cases when nominal MPC leads to ISS – Case 3 [2]

There are special nominal MPC formulations with state constraints that provide ISS.
Intuition: Formulation has to be such that state constraints are not active in a robust invariant set, where the system is then ISS.

C3: If the system has constraints on the states and inputs ($x \in \mathcal{X}, u \in \mathcal{U}$), $l_f(x) := \lambda V(x)$ where $V(x)$ is a uniformly continuous Lyapunov function in $\mathcal{X}_f$ and $\lambda \geq 1$, and $V_N(x, U)$ is uniformly continuous in x for all $x \in \mathcal{X}_N$ and $U \in \mathcal{U}^N$, then there exists a region $\Omega_r = \{x \in \mathbb{R}^n \mid V_N^\star(x) \leq r\} \subset \mathcal{X}_N$ in which the optimal cost is uniformly continuous.
Proof idea:

- For all $x \in \Omega_r$, the predicted optimal trajectory can be shown to remain in Ω_r and hence the state constraints are not active.
- The particular choice of the terminal cost allows to show that there exists a $r(\lambda)$ such that for all $x \in \Omega_r$ the terminal constraint is not active throughout the state evolution. The terminal constraint could also be removed from the optimization problem.[2]

$\rightarrow$ Set of constraints of nominal MPC problem does not depend on state x, apply arguments of problem with only input constraints.

[2] Limon, D., Alamo, T., Salas, F., Camacho, E.F., "On the stability of constrained MPC without terminal constraint". IEEE Trans. on Automatic Control, 51, 832-836, 2006.

Remarks

- If instead of uniform continuity in the previous results, only local continuity in a neighborhood of $x = 0$, $w = 0$ can be shown, the results can be adapted to show local ISS.
- Extensions for suboptimal MPC solutions exist, if the optimal MPC cost is continuous.
- Various generalizations under reduced assumptions exist (e.g. avoiding certain continuity assumptions).

MPC for bounded uncertainties - Question 2

Uncertain constrained linear system
$$x(k+1) = f(x(k), u(k), w(k)) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all disturbance realizations **if possible**
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Question 2:
What happens if we ignore the noise and apply the nominal MPC control law relaxing the state constraints?
Can we achieve these properties with nominal MPC **while enlarging the region of attraction**?

Soft-constrained nominal MPC controller

$$
\begin{aligned}
V_N^{\text{soft}}(x(k)) = \quad & \min_{U,\epsilon_0,\ldots,\epsilon_N} \quad l_f(x_N) + l_\epsilon(\epsilon_N) + \sum_{i=0}^{N-1} l(x_i, u_i) + l_\epsilon(\epsilon_i) \\
& \text{subj. to} \\
& \qquad\qquad x_{i+1} = \bar{f}(x_i, u_i),\ \ i = 0, \ldots, N-1 \\
& \qquad\qquad H_x x_i \le c_x + \epsilon_i,\ \ i = 0, \ldots, N-1 \\
& \qquad\qquad u_i \in \mathcal{U},\ \ i = 0, \ldots, N-1 \\
& \qquad\qquad H_N x_N \le c_N + \epsilon_N \\
& \qquad\qquad x_0 = x(k)
\end{aligned}
$$

- l_ϵ is a penalty on the slack variables, e.g. $l_\epsilon(\epsilon) = v_1\|\epsilon\|_1 + v_2\|\epsilon\|_2^2$
- $\mathcal{X}_N^{\text{soft}}$ denotes the enlarged feasible set

$\rightarrow$ No hard state constraints

$\rightarrow$ But: $V_N^{\text{soft}}(x)$ is not a Lyapunov function in $\mathcal{X}_N^{\text{soft}}$ without further assumptions.

Soft-constrained MPC with ISS properties

Examples from the research literature:

- Re-define slack on terminal constraint (shown for linear systems)
  M. N. Zeilinger, M. Morari and C. N. Jones, "Soft Constrained Model Predictive Control With Robust Stability Guarantees," in IEEE Transactions on Automatic Control, vol. 59, no. 5, pp. 1190-1202, 2014.
  K. P. Wabersich, R. Krishnadas and M. N. Zeilinger, "A Soft Constrained MPC Formulation Enabling Learning From Trajectories With Constraint Violations," in IEEE Control Systems Letters, vol. 6, pp. 980-985, 2022.

- Contractive terminal constraint
  S. Yu, M. Reble, H. Chen, F. Allgöwer, "Inherent robustness properties of quasi-infinite horizon nonlinear model predictive control" in Automatica, vol. 50, no. 9, pp. 2269-2280, 2014.

- Relaxed barrier function as slack penalty, re-define terminal cost including slack penalty (shown for linear systems)
  Ch. Feller, Ch. Ebenbauer, "A stabilizing iteration scheme for model predictive control based on relaxed barrier functions", Automatica, vol. 80, pp. 328-339, 2017.

$\rightarrow$ Requires special soft-constrained formulation and/or assumptions.

Nominal MPC for Uncertain Systems - Summary

Idea

- Ignore the noise and hope it works

Benefits

- Simple
- No knowledge of the noise set $\mathcal{W}$ required - 'just works'
- Often very effective in practice (this is what most practitioners do anyway)
- Feasible set is larger (we can find a solution, but it may not work)
- Region of attraction may be larger than other approaches

Cons

- Works for linear systems, for nonlinear systems only under stronger continuity assumptions (in absence of state constraints, or specific formulations)
- Very difficult to determine region of attraction (set of states in which the controller works)
- Hard to tune - no obvious way to tradeoff robustness against performance

Literature

1. J. B. Rawlings, D. Q. Mayne, M. M. Diehl (2017). Model Predictive Control: Theory, Computation, and Design. Nob Hill Publishing, 2nd edition.
2. Limon, D., Alamo, T., Raimondo, D. M., Muñoz de la Peña, D., Bravo, J. M., Ferramosca, A., and Camacho, E. F. (2009). "Input-to-State Stability: A Unifying Framework for Robust Model Predictive Control", In L. Magni, D. M. Raimondo, & F. Allgöwer (Eds.), Nonlinear Model Predictive Control, vol. 384, pp. 1-26, Springer Berlin Heidelberg.
3. Jiang, Z.-P.,Wang, Y. (2001). "Input-to-state stability for discrete-time nonlinear systems", Automatica, 37, 857-869.
4. A. R. Teel (2004). "Discrete time receding horizon control: is the stability robust", In Marcia S. de Queiroz, Michael Malisoff, and Peter Wolenski, editors, Optimal control, stabilization and nonsmooth analysis, volume 301 of Lecture notes in control and information sciences, pages 3-28. Springer.